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Studies concerning the curriculum, the child, the learning environment, and teaching methods are covered in the four parts of this guide to current research in the elementary school mathematics. Subjects of the first part include the sources of the curriculum, the relationship of Piaget's work in child development to achievement in mathematics, Cuisenaire materials, innovative programs, kindergarten programs, and comparison of U.S. and foreign programs. Readiness, conceptualization, and achievement of normal, gifted, mentally retarded, and culturally deprived children are discussed in Part II. Their attitude, anxiety, emotional disturbance, personality, and self-concept in relation to mathematical learning are also considered in Part II. Part III explores class size, groups, time allotment, textbooks, teacher training, and inservice education. Approaches to instruction, motivation, diagnosis, mental growth, programed instruction, and methods of teaching segments of mathematics are the subjects considered in Part IV. (DO)

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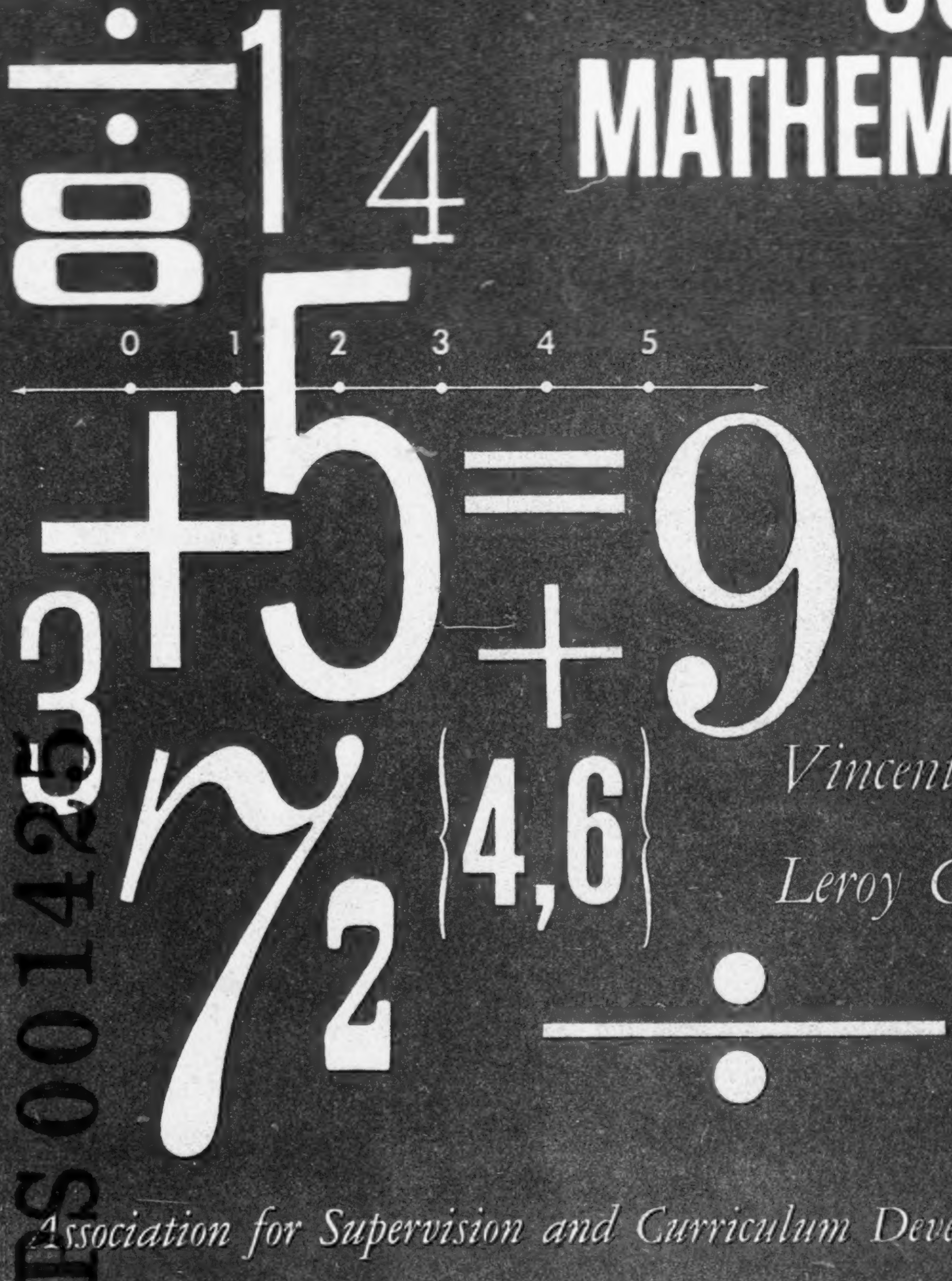
Guide to Current Research

ELEMENTARY SCHOOL MATHEMATICS



Vincent J. Glennon
Leroy G. Callahan

Association for Supervision and Curriculum Development, NEA

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ELEMENTARY SCHOOL MATHEMATICS

A Guide to Current Research

Third Edition

Vincent J. Glennon
Professor of Education
Syracuse University
Syracuse, New York

Leroy G. Callahan
Assistant Professor
Faculty of Educational Studies
State University of New York at Buffalo
Buffalo, New York

PS 001425

Association for Supervision and Curriculum Development, NEA
1201 Sixteenth Street, N.W., Washington, D.C. 20036

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Foreword

Perhaps the readers of ASCD publications are best characterized as those who affect curriculum decisions. Attempts to answer such perplexing questions as "What should be taught?", "When should it be taught?", and "How should it be taught?" give rise to many lesser questions requiring both value orientations and facts. We are cautioned against quick, easy answers to these difficult questions. Such answers are not to be found, but there is real advantage to short, succinct presentation of the facts and authoritative opinions bearing on them. *Elementary School Mathematics: A Guide to Current Research* asks the right questions about the teaching of mathematics in elementary schools and very briefly reports the best of current research related to these questions. When the research evidence is not appropriate or available, an authoritative point of view and the kernel of the rationale are presented.

You may recall two earlier ASCD publications in 1952 and 1958, each titled *What Does Research Say About Arithmetic?* Extremely rapid and significant developments in this field during the past decade required both a new publication and a new title reflecting a broader emphasis than arithmetic. Thus this booklet joins *Elementary School Science: A Guide to Current Research,* by Maxine Dunfee, and *Improving Language Arts Instruction Through Research,* by Harold Shane and June Grant Mulry as one of a continuing series devoted to the application of research findings to curriculum decision making.

Vincent Glennon and Leroy Callahan did not write this booklet for the mathematician or for the specialist in mathematics education, though both may gain a different perspective from it. Neither is it written for the layman who has little concern with many of the professional questions to which it is addressed. It is instead for the educator who finds mathematics education one among many of his professional concerns. Such an educator will find help here with many of the questions he has about the mathematics programs of elementary schools and may be led to several of the nearly three hundred original sources that are cited.

May 1968

J. Harlan Shores, *President 1967-1968*
Association for Supervision and
Curriculum Development

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The Association wishes to express its appreciation to Vincent J. Glennon and Leroy G. Callahan for their painstaking care in surveying and reporting the research for this study, for the writing and editing of the manuscript, for preparing the index, and for cooperating in the technical production of this booklet.

Final editing and production were the responsibility of Robert R. Leeper, Associate Secretary and Editor, ASCD publications. Technical production was handled by Claire J. Larson assisted by Joan H. Steffenburg and Mary Ann Lurch.

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Introduction

This is the third edition of this research monograph. The prior editions were published by the Association for Supervision and Curriculum Development in 1952 and 1958, and each went through several printings. It is gratifying to the authors to know that both the content and the method of organizing and presenting the content to the persons interested in mathematics education have continued so well to meet their needs.

The earlier editions were entitled "What Does Research Say About Arithmetic?" The new title evidences the authors' concern for the broader concept of mathematics in the school program which, although it had earlier roots as far back as the mid 1930's, received an added impetus and general acceptance after 1958. Today the mathematics program of the elementary school is rarely referred to as arithmetic, since this term no longer accurately or adequately denotes the metes and bounds of the program.

As with the prior editions, the purpose of this edition continues to be an attempt by the authors to identify the salient questions of concern to school personnel, to search the literature for worthwhile studies, and to summarize these studies in such form that the use thereof can make a difference in the classroom.

Also, as with the prior editions, the reader should keep in mind the following points:

1. Although most of the answers to the questions are research-based, there are many questions of importance to school personnel that have not been researched in an empirical way but are included in the monograph. School personnel must make wise decisions on many important questions in the absence of hard research evidence. The authors have attempted to present well-balanced summaries of the several facets of each of these questions.

2. It is not possible to summarize the findings of all available research and philosophical discussions in a monograph of this size. Choices had to be made. The reader is urged to consult the original sources for more complete discussions of the questions.

3. Although school personnel can feel secure in teaching along the

lines suggested in the monograph, they should recognize that the answer to any question is subject to modification in the light of subsequent research and other non-empirical types of studies.

The authors wish to make it clear that full responsibility for the accuracy of interpretation of the studies and valid representation of the studies in the paragraphs quoted rests with themselves.

Appreciation is cordially expressed to Dean David R. Krathwohl for appointing Dr. Callahan to a post-doctoral research and teaching lectureship in the School of Education, Syracuse University, during the 1966-67 academic year, to Dr. C.W. Hunnicutt, co-author of the first and second editions, for his continued participation in and enthusiasm for the project over many years, and to Conrad Campbell for assisting with the writing of several parts of the manuscript.

The authors also express appreciation to Mrs. Muriel Bitensky and Mrs. Ana Boneo for typing the manuscript.

May 1968

VINCENT J. GLENNON
LEROY G. CALLAHAN

We are only just realizing that the art and science of education require a genius and a study of their own; and that this genius and this science are more than a bare knowledge of some branch of science or literature.[1]

ALFRED NORTH WHITEHEAD (1929)

. . . it is true of arithmetic as it is of poetry that in some place and at some time it ought to be a good to be appreciated on its own account—just as an enjoyable experience, in short. If it is not, then when the time and place come for it to be used as a means or instrumentality, it will be in just that much handicapped. Never having been realized or appreciated for itself, one will miss something of its capacity as a resource for other ends . . .[2]

JOHN DEWEY (1930)

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[1] Alfred North Whitehead. *The Aims of Education and Other Essays*. New York: The Macmillan Company, 1929. p. 6.

[2] John Dewey. *Democracy and Education*. New York: The Free Press, 1930. p. 240.

Part One

Studies Concerning the Curriculum

What are the main sources of the mathematics curriculum?

The elementary school mathematics curriculum, like all other subject areas that make up general education as distinguished from specialized or vocation-oriented education (the former concerned with living, the latter with earning a living), is derived from three sources. Like the farmer's three-legged milking stool, the curriculum is a well-balanced, stable instrument if and only if the three sources contribute to it equally.

The three sources of the elementary school mathematics curriculum may be referred to as the nature of the learner, the nature of his adult society, and the nature of the cognitive area—mathematics. The first of these may be referred to as the expressed needs-of-the-child theory of curriculum, or the psychological theory. The second, as the needs-of-adult society, social utility, instrumentalism, or sociological theory of curriculum; and the third, as the structural, the pure mathematical, or the logical theory of curriculum.

Each has something to contribute to a well-designed curriculum. Each theory has its strong supporters and its equally strong opponents; and in each group are some people who are quite unaware that there are any other points of view than their own. Any unilateral authoritarian view of the curricular basis of the program is an extremist view. In order to have a clear perception of a balanced theory of curriculum, one must first have a clear perception of each of these extremist theories. Each is discussed briefly below.

1. *The psychological basis for curriculum theory.* The question of what mathematics is of most worth to elementary school children can be viewed from two quite different psychological approaches. One approach can be labeled the cognitive-developmental point of view, the other the clinical-personality point of view. Neither point of view is clearly self-contained; each may draw upon the other to varying degrees

depending upon the biases in the professional training of the person doing the viewing.

The cognitive developmental approach to curriculum theory emphasizes the nature of the subject matter being learned and the developmental stages in the learning. The examplars of this point of view in the world today are Jean Piaget in Switzerland, and, in this country, William A. Brownell.

The clinical-personality point of view emphasizes the affective aspect of human development. The extremist point of view is best evidenced in the work and the writing of A. S. Neill and particularly in *Summerhill: A Radical Approach to Child Rearing* (187). In the entire book of almost 400 pages, the arithmetic curriculum and the methods of teaching it are referred to in only six very brief statements. In essence, Neill dismisses as irrelevant or inappropriate any efforts on the part of teachers or parents to preplan a program in elementary school mathematics. In a word, he is of the opinion that the only honest source of the content, of methods, of learning materials, and of the evaluation, too, must and can only eventuate out of the needs of the child as he expresses them.

2. *The sociological basis for curriculum theory.* Those who advocate a sociological approach only to the selection of the content of the elementary school mathematics program are of the opinion that the only worthwhile mathematics is that which has previously been judged of great usefulness to the average adult in business situations and in general life situations. Mathematical topics which do not meet a rigorous interpretation of this criterion, they argue, are not a legitimate part of the general education of the child. Such topics, therefore, become a part of the specialized or vocational education of the older child or young adult to be learned in a vocational program either in the school or on the job.

Over a fifty-year span of professional activity beginning about 1911, Guy M. Wilson and his students have done the greatest amount of research on the question, "What mathematics is important enough in business and life as to be the *mastery* program in the elementary school?" (287). Wilson answers the question succinctly in these words:

> This question can be answered quite specifically and authoritatively on the basis of curricular studies as to the usage of arithmetic in business and life. It is no longer necessary to rely upon guesswork or mere opinion. This question of essential drill (for mastery) will be discussed again and again in connection with topics of arithmetic, but here it may be noted that the drill material for mastery consists of simple addition—100 primary facts, 300 decade facts, carrying and other process difficulties; simple subtraction—100 pri-

mary facts, process difficulties; multiplication—100 primary facts, process difficulties; multiplication—100 primary facts, process difficulties; long division —no committed facts, general scheme and process steps; simple fractions in halves, fourths, and thirds, and, in special cases, in eighths and twelfths, general acquaintance with other simple fractions; decimals—reading knowledge only.

The essential drill phases of arithmetic for perfect mastery are as simple as that. The load is very small . . . (pp. 3, 4).

Figure 1. Showing a summary of all the functions of the Dalrymple Study in terms of denominators

(1) Units	(2) Different fractions	(3) Sears-Roebuck Co. study	(4) Boston Transcript study	(5) United Drug Co. study	(6) Jordan Marsh Co. study	(7) Hotels study	(8) S. S. Pierce Co. study	(9) A. Stocks, bonds, etc.	(10) B. Bank rates	(11) C. Foreign exchange	(12) Total	Percentage
Halves	1	9,069	12,751	2,212	259	110	327	6,175	731	242	31,876	31.184
Quarters	2	6,671	12,717	956	177	29	89	2,624	12,911	296	36,470	35.678
Thirds	2	803	....	110	120	15	1	....	....	......	1,049	1.026
Fifths	4	18	....	4	....	......	1	....	....	......	23	.023
Sixths	5	2	....	100	11	8	1	....	....	......	122	.119
Eighths	7	2,555	12,880	267	1,174	......	9	3,665	....	268	20,818	20.368
Ninths	7	1	....	4	1	......	......	....	....	......	6	.006
Tenths	9	23	....	4	....	......	......	....	....	......	27	.026
Twelfths	11	4	....	76	843	46	......	....	....	......	969	.948
Fifteenths	5	....	....	2	....	......	......	....	....	......	2	.002
Sixteenths	14	682	....	154	....	4	2	....	....	101	943	.923
Twentieths	2	4	....	1	....	......	......	....	....	......	5	.005
Twenty-fourths	4	4	....	....	....	......	1	....	....	......	5	.005
Thirtieths	2	....	....	2	....	......	1	....	....	......	3	.003
Thirty-seconds	31	148	255	4	....	......	......	9,260	10	......	9,677	9.467
Thirty-sixths	4	4	....	....	....	......	......	....	....	......	4	.004
Fortieths	1	....	....	....	....	......	1	....	....	......	1	.001
Forty-eighths	1	1	....	....	....	......	......	....	....	......	1	.001
Fiftieths	1	....	....	....	....	......	1	....	....	......	1	.001
Sixtieths	1	....	....	....	....	......	1	....	....	......	1	.001
Sixty-fourths	8	74	....	....	....	......	......	....	....	......	74	.072
Hundredths	27	133	....	....	....	......	......	....	....	......	133	.130
$3/144$	......	3	....	....	....	......	......	....	....	......	3	.003
$925/1000$	......	1	....	....	....	......	......	....	....	......	1	.001
$3/4$	......	6	....	....	....	......	......	....	....	......	6	.006
Totals	149	20,199	38,603	3,896	2,585	212	435	21,724	13,652	907	102,220	100.00

Columns 3 to 11 of Figure 1 show interesting variations in fractions used in different lines of business. The fraction *one-half* occurs with reasonable frequency in all units. The same is true of *fourths*. *Thirds*, on the other hand, do not appear under the *Boston Transcript* unit, which is chiefly a summary of stock-market quotations.

The reader who is unacquainted with this extremist point of view might well ask how the data were gathered and collated to form the basis for curricular decisions. The table on the preceding page shows a classification of fractions (rational numbers named by fractions) used in business situations and gathered over a two-year period in the Boston area. (See Wilson, Table 1, p. 201.)

On the basis of this and similar studies, Wilson concludes that fractions "used in business are much simpler than the fractions (taught) in the schools. It may be remarked also . . . that the operations and combinations of fractions in business are very, very simple in comparison with school practices."

He asks, "Is it possible that we have been wasting much school time on useless fractions? And in going beyond usage on a purely manipulative basis, have we not done much to confuse and defeat the child?"

3. *The logical, or pure mathematical, basis for curriculum theory.* The third source of the curriculum, or the third leg of the farmer's milking stool, to continue our simile, is usually named the logical, or structural, or pure mathematical source. Extremists who hold this point of view exclusively are usually trained as mathematicians and have little insight into or concern for the points of view held by the groups representing either a psychological approach or a sociological approach. Their main concern is that of transmitting the mathematics in a form uncontaminated or undefiled by any relating of the pure structure to socially useful situations. By way of an illustration, if a fifth grade group of children studying about Mexico and its people were learning or using, or both, the cognitive capability of multiplying a fraction by 2 in order to double the amount of some ingredient used in tortillas, this experience would be denigrated by referring to it as "some sort of home economics perhaps but certainly not mathematics."

The historical roots of the sociological theory of curriculum are as old as early human attempts at transmitting the customs of the tribe to the young; and the historical roots of the psychological theory can be found in the writings of Pestalozzi, Herbart, Froebel, and more recently Freud, Adler, Jung, and the cognitive-developmental psychologists, G. Stanley Hall, William James, Charles Hubbard Judd, William A. Brownell, and Jean Piaget. But the historical roots of the "pure mathematical" theory of curriculum can be traced back at least 2,500 years to the beginning attempts of the Greek mathematicians to structure the subject. Substantial contributions to the purification process were made in the past few hundred years by DeMorgan, Hamilton, Peano, and others.

As a consequence of the work of these men and of the abstract nature of the subject, no cognitive area has as elegant a structure as mathematics. Whereas in any subject matter area in the social sciences, say geography, we might list an almost endless set of principles, in the real number system we have only 11 "principles" (properties) or axioms, and 3 equality axioms.

In part, the recent efforts to purify the elementary school math program may be due to the fact that some few mathematicians, enamored with the elegance of the subject, want all others to see the beauty of the abstract structure as they see it, and in so wanting, press vigorously for the widespread adoption of the pure mathematics approach as the only legitimate theory of curriculum.

4. *Balance among the three theories.* The authors have found it useful in attempting to help school personnel "make sense" out of the ebb and flow of curriculum change to use a model of a triangle to picture the extreme points of view.

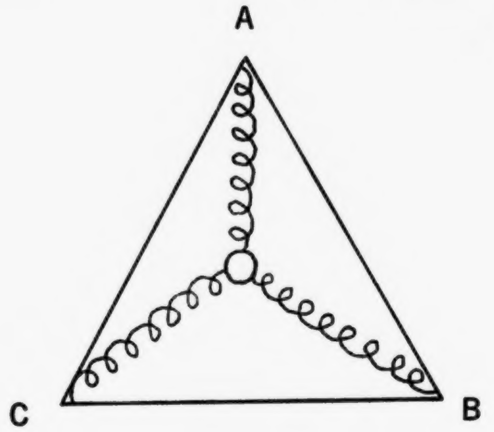

Figure 2.

Each of the three extremist positions can be viewed as one of the vertices of the triangle. A balance among the three theories can be pictured as a ring held in place by springs each fixed in place at a vertex. The pressures of society on the school curriculum in our century have caused the center of balance to shift often in our century alone. Professional education was interpreted by some child developmentalists as a powerful spring which pulled the center of balance toward point A.

Pragmatism, with its implications for a socially useful curriculum, was viewed by some extremists as sufficient cause to justify pulling the center of balance toward point B. And now, the concern for the logical structure, the purity of the mathematics, since 1957 has caused some extremists to argue for a shift in curriculum toward point C.

In each instance, the more extreme the position of a person or the program he advocated the more it moved from a central position toward one of the three vertices. The curriculum approach implemented by A. S. Neill noted above represents the most radical extremism toward point A, or *at* point A. The curriculum innovations of Guy M. Wilson represent the farthest deviation toward point B. And certain innovations since 1957 which are concerned with mathematics for its own sake in the elementary grades represent the greatest distortion of the curriculum in a move toward point C.

The difficulty of obtaining and maintaining balance among the extremes is discussed by Foshay (98):

> A conception of what balance means in the curriculum is a necessity in any time. In these days of upheaval in education, however, such a conception is an urgent necessity. It is possible that the new curriculum patterns, when they have emerged, will prove to be in better balance than anything we have known. However, taken as a whole, it could be that the new curriculum will imply a distorted version of our culture, or our ideals as a people, even what we want an American to be. This has happened in the past, at those times when it has become apparent that the existing curriculum no longer fits the times. The changes have not always proved to be improvements; sometimes, despite the best efforts of wise men, the result has been only to substitute one distortion for another.

Do the innovative programs possess curricular face validity?

The primary question to ask about any school program, whether contemporary or innovative, whether the product of a group effort or of an individual's effort, whether in mathematics or any other cognitive area, is, "Is this program valid from a curricular (not statistical) point of view?" That is, is the content of the program analyzed by responsible, open-minded, well-balanced, and mature professional workers, judged to be good and appropriate for the children for whom it is intended?

To answer this question about innovative mathematics programs for children, the National Council of Teachers of Mathematics appointed a committee of five people who, assisted by subcommittees, prepared the report entitled *An Analysis of New Mathematics Programs* (5). The committee "agreed upon a set of eight issues which seemed to be crucial. . . ." The eight issues are:

1. How much emphasis should be placed on the social applications of mathematics? What should be the purpose and nature of these applications?

2. At a particular grade level, what topics can be most effectively developed and which are most appropriate?

3. What emphasis should be placed on the study of mathematical structures to bring about a better understanding and use of mathematics?

4. How rapidly should the student be led from the use of the general unsophisticated language of mathematics to the very precise and sophisticated use of it?

5. What is the relative merit of presenting a sequence of activities from which a student may independently come to recognize the desired knowledge as opposed to presenting the knowledge and helping students rationalize it?

6. What relationship should exist in the mathematics programs between the function of developing concepts and that of developing skill in the manipulation of symbols?

7. At what level should proof be introduced and with what degree of rigor? How rapidly should a student be led to make proofs independently? At what level should he be aware of what mathematical proof is?

8. Are there available measures of the changes taking place that can be applied at this time, and what provisions can be made for evaluating the same changes in the future?

The committee applied these criteria of curricular validity to eight innovative programs. Of the eight programs *three* were prepared in whole or in large part for the elementary school grades and are within the concern of this research monograph. We are restricting our presentation to the first criterion, "Social Applications," trusting that the interested reader will consult the original report.

1. *The Greater Cleveland Mathematics Program*

Social Applications: There are more than 200 problems related to social applications in the first grade material and more than 800 in the second and third grade material. Approximately 9 percent of the problem material in the first, second, and third grades is devoted primarily to social applications. Of the 179 statements of objectives, 28, or 16 percent, deal directly with social applications.

In addition to the material listed above, there are 120 story problems for the second grade and 228 story problems for the third grade dealing with concepts and skills taught in the units in which they are presented. The story problems represent approximately 3 percent of the total problem material.

2. *The Madison Project*

Social Applications: The Madison Project material has few social applications. For example: in *Discovery of Algebra,* Chapter 1, "Equations, Open Sentences, and Inequalities," there are only five examples; one of these topics is the theory of gravitation, which is unfamiliar to the pupils. Social applications dealing with the stock market, robbery, and stolen money are also poor. In all there are about 50 verbal problems, other than those which children are asked to make themselves.

3. *School Mathematics Study Group* (S.M.S.G.)

Social Applications: Sections of the reviewed material give considerable attention to social application. This can be found in some of the exploration questions, in references to historical development, and certainly in the problems contained in the units involving the four basic operations. But, in general, this material gives limited attention to social usage. The potential for social application was evident in much of the content, and attention could have been given to the aspect of social usage without sacrificing the mathematical integrity.

Many of the problem exercises were highly suggestive of social application. However, it was clear that the problems posed were placed there primarily to demonstrate the application of a process or a concept. Relatively few of the problems posed were the kind which made the learner aware of the need for a new mathematical process and the concomitant conceptual development. Purpose, which becomes apparent to the user through the social application of his mathematical learnings, seemed often neglected.

In the units on linear measurement, angle measurement, and recognition of common figures, some references to everyday usage were considered. But here, too, such consideration was sparse in terms of the development of the purely mathematical aspects. The three units devoted to the theory of sets were just that, and the situations contrived were concerned with operation within the theory rather than with social application.

Other evaluative studies of the appropriateness and worthwhileness of innovative programs have been made by groups and individuals. One of the more noteworthy is the Cambridge Report (112). A few evaluative statements *pro* and *con* are summarized below.

THE CAMBRIDGE REPORT:

In the Foreword, Keppel, then United States Commissioner of Education, states the case *for* the Report as follows:

> The present report is a bold step toward meeting this problem (the lag in education). . . . the report simply states (their) views as to the direction in which we should now be going. They have set forth goals for the future simply so that we may have some informed notion of the steps we should be taking, right now, if we are ever to make real progress.

Marshall Stone, professor of mathematics, reviewed the report with very different evaluative statements. Among these are the following:

> . . . the Cambridge Report is extremely disappointing. . . . it is really superficial, confused, and shot through with wishful thinking of a shallow kind. It is superficial in its treatment of the content and organization of the mathematics curriculum; confused in the presence of the very deep reasons for a new, modern approach to algebra and geometry at the school level, and willful in its refusal to face up to the pedagogical difficulties involved in the sweeping changes it proposes.

> . . . It is a terrible weakness in the Cambridge Report, therefore, that it does not ask, "what mathematics should we teach for *whom*?"

The above question, "Do the innovative programs possess curricular validity? was concerned with the judged curricular validity (the appropriateness, the worthwhileness) and was answered in the light of the eight stated criteria. Another approach to judging the worthwhileness of an innovative program, assuming, of course, that it is first judged to have curricular validity, is by way of an experimental study. Here we would be concerned with finding answers to the question, "Does Innovative Program X result in better learning (or more learning, or greater time saved, etc.) than Contemporary Program Y?"

There is very little valid and dependable evidence from studies with experimental designs. Very few studies have been done, and very few of those stand up under close scrutiny using the criterion of "common sense." Brownell (39) discusses general weaknesses found in many experimental studies.

> It is, I think, highly doubtful that the quality of evaluative research will be enhanced as much by new technical improvements as by (common sense). This is not to say that we should call a moratorium of efforts directed to technical advances. Quite the contrary: the search for new and better means of controlling experimental factors and of treating quantitative data should certainly be encouraged; and the evaluator will do well to employ all proven improvements in his own investigations. Even so, such gains provide no guarantee that his research will consequently be much sounder and more fruitful.
>
> Let me illustrate what I have in mind. Over the years there has been commendable progress in statistical methods of ascertaining the reliability of differences between the means of test scores of experimental groups. This progress, however, has not been matched by equal progress in our methods of procuring valid and relevant data to start with or by our methods of determining the *educational* significance of differences that stand up statistically. Thus, even the most refined formulas for determining the reliability of differences do not eliminate the possibility of drawing conclusions that are false or otherwise indefensible as far as educational practice is concerned.

And Theodore R. Sizer (241), Dean of the Harvard Graduate School of Education, also discusses his concern about the dangers and problems in the curriculum reform movement.

> A fourth problem that seems to bedevil us is what I like to call the "gee whiz disease." There is nothing more exciting than seeing a child . . . reaching an understanding which he has never had before. . . . Too often curriculum development has been so overwhelmed by this phenomenon that the "development" seems to consist simply of a series of experiments designed to give the instructor his "gee whiz" kick. The kids like this, and the teachers too, but there are more profound questions that we all should be asking.

> We cannot build a curriculum on top of a series of relatively unconnected experiments which lead to a supposed learning. The "gee whiz" reaction simply is not enough.
>
> I . . . am saying that we need intellectual honesty and humility as well as enthusiasm. We cannot be carried away by the inherent excitement of the process of education and the process of teaching. Curriculum developers and the children they teach deserve better.

What does the work of Piaget suggest about the cognitive development of the child?

Since the last revision of this monograph there has been a proliferation of studies and writings on the developmental psychology of Jean Piaget. It is beyond the scope of the monograph to attempt a comprehensive analysis of Piaget's work, which has evolved over a period of time greater than 40 years. The reader interested in a comprehensive analysis of Piaget's work would do well to begin with Flavell's (97) work on this topic. It does seem necessary, however, to attempt a thumbnail sketch of Piaget's developmental theory if we are to look intelligently for implications for the teaching of elementary school mathematics.

Development, to Piaget, is conceived as the rather gradual adaptation of the child to his environment. Adaptation is a dynamic process of the individual in his attempt to attain equilibrium between two complementary processes: assimilation and accommodation. The process of adaptation requires both an incorporation of the external world, assimilation, and a transformation of the internal world of the child, accommodation. As Rosenbloom (217) puts it, "assimilation" is the incorporation of the objects into the child's patterns of behavior, the changing of the signals the child receives from his environment to fit the mental structures he already has. "Accommodation" is the modification of the child's patterns of behavior to fit his environment, the changing of his mental structures to fit the signals he receives from his environment. If a child calls a cloud a bear, he is "assimilating" his perceptions to the mental structures he already has. When he learns to classify clouds as nimbus or cumulus, he is "accommodating" his mental operations to his perceptions.

Piaget has studied various components of external reality in order to more clearly comprehend the child's progressive adaptation to his environs. These components include: space and time, external objects, and causality. For each one of these components that have been studied, development is reflected in a series of successive levels or stages. These stages are:

I. Sensori-Motor intelligence (birth—4)
II. Intuitive thought (4—7)
III. Concrete operations (7—11)
IV. Formal operations (11—15).

The indicated age ranges are only approximations in our culture. These ranges will probably differ greatly between individuals.

Kesson (159) gives an example of progression through these stages using the idea of a mathematical function. He writes:

> . . . in every one of these (stages) the child may be said to have some kind of "function" concept. In the very earliest period of sensori-motor development, physical movements are operators on certain events in the environment resulting in a new set of correspondences. The child moves objects in and out of his perceptual field; this, even in the absence of a notion of the permanent object. Piaget is quite clear in his consideration of the sensori-motor schemata. He calls them, "instinctive premathematical structures." At a second level of development, the preoperational one (intuitive), the child does not recognize the reversibility of displacements. . . . He has some primitive idea of the notion of a function—in his real life, take the case of the gum machine in which a domain of pennies is mapped onto the range of gum balls —but he has none of the axioms of a function theory.
>
> At the next stage—that of concrete operations—the child is able to do most of the things that older children and adults are able to do but only in the presence of concrete materials. He can solve problems, indeed quite complex problems of combinations and of physical space, but only with particular materials. It is in the last stage of logical operations, which Piaget dates from 11 years or so, that the child finally achieves "liberation of the logical mathematical structures from their dependence on experience." All that has gone before this liberation is there; the child does not begin anew but only now can his thought be free of the concrete experience.

Let us now look at a few specific examples from Piaget's studies that may have some direct implications for teaching elementary school mathematics. One object of inquiry has been the conservation of number. Piaget (201) writes:

> Our contention is merely that conservation is a necessary condition for all rational activity . . . . This being so, arithmetical thought is no exception to the rule. A set or collection is only conceivable if it remains unchanged irrespective of the changes occurring in the relationships between elements.

For example, a child is said to conserve number when he realizes that the numerical equality between two collections of objects remains unchanged following a change in the spatial arrangements of the objects, provided no objects are added or taken away.

Generally, Piaget has found that children go through the following stages in the development of conservation of number: up to 5 years, no conservation; 5-6 years, a phenomenistic, unstable sort of notion of conservation based on the observation and manipulation of different forms and transformations; 6-7½ years, a logical, axiomatic certainty of conservation in the case of all transformations.

Another object of inquiry involved the conservation of length. The importance of this concept is pointed up by Piaget (202) when he writes, "Underlying all measurement is the notion that an object remains constant in size throughout any change in position."

This type of experiment consisted in showing the children two straight sticks identical in length and with their extremities facing each other; one of the sticks was then moved forward 1 or 2 cm. and the subject was asked to say once again which of the two was longer or whether they were the same length. At all levels, the sticks were judged equal before staggering. After that change of position, subjects at the first stage (4-5 years) maintain that the stick which has been moved forward is longer, thinking only in terms of the further extremities and ignoring the nearer extremities. Between stages IIA and IIb (5-6 years) we find a series of transitional responses (unstable, based on observation and manipulation). The child then moves (6-7½ years) to an operational level where conservation of length is assured.

The attempt in this section has been to give a very short overview of Piaget's developmental theory and to give some specific examples of his objects of inquiry that may have significance for teachers of elementary school mathematics. In the following sections some specific questions regarding Piaget's work will be examined as it pertains to the teacher of elementary school mathematics.

How fixed is the rate of progress through Piaget's stages?

If one accepts the idea that the work of Piaget has implications for the teaching of elementary school mathematics, two related questions present themselves: (a) How fixed is the rate of progress through these stages? (b) Given a negative response to question one (negative in the sense that the rate is not maturationally fixed), how can the teacher most efficiently and effectively guide students to the highest levels of logical and symbolic thinking?

In answer to the first question, an increasing reservoir of experimental replications of Piaget's initial observations would seem to indicate that the rate of progress through the stages can be accelerated or retarded depending upon the experience of the individual. Piaget (203), himself, writes on this matter that:

> . . . Progressive construction does not seem to depend on maturation, because the achievements hardly correspond to a particular age. Only the order of succession is constant. However, one witnesses innumerable accelerations or retardations for reasons of education (cultural) or acquired experience.

Sister Gilmary (106), after reviewing the experimental replications dealing with Piaget's work in the United States, Canada, Britain, and Scandinavia, points out that there is "accord with Piaget's earlier interpretation of stages of mental structuring." However, they have also found that the specific stages and levels of operation are neither rigid nor mutually exclusive, and are generally applicable to children at much earlier ages than Piaget so judged.

Almy's (3) longitudinal and cross-sectional studies of the emergence of conservation of number and amount of liquid, using children from two kinds of cultural background, point up the importance of experience in the development of Piaget's stages. Using three conservation tasks that were ranked by order of difficulty, 58 percent of the students from a lower class school were unable to conserve in any but the easiest task by the time they were in the second grade. This was a larger proportion than was the case with children from a middle-class school when they were in kindergarten. The sequence of progress, however, appeared to be very similar for the two groups during the 3 years (K, 1, 2) that their growth in mental development on these tasks was studied longitudinally.

The teacher wanting to use the findings of Piaget and his followers can feel quite confident that the sequence of stages delineated may offer a useful structure for making curriculum and teaching method decisions. The rate at which children progress through the stages, however, will be affected by many factors and will vary among tasks and also across cultural lines.

What relationship exists between growth in Piaget's tasks and achievement in arithmetic?

Almy (3) was concerned with the general question, "To what extent does knowledge of a child's progress in conservation provide a teacher with information relevant to his ability to cope with the tasks posed in the classroom?" Two related studies were conducted, one cross-sectional and one longitudinal. One group of children was drawn from a middle-class neighborhood, another from a lower-class neighborhood. In the cross-sectional study, measurements were taken at the kindergarten, first, and second grade levels. The longitudinal study followed the same kindergarten students used in the cross-sectional study through the second grade.

Three Piagetian conservation tasks were given to these students each along with other tasks of achievement and mental ability. Of concern here is the mathematics achievement test that was administered to the students. This test, "New York Inventory of Mathematical Concepts," has two sections; one dealing with "premeasurement," the other with "numerical concepts." Correlations between the numerical concept part and ability to conserve on the Piagetian tasks were found to be 0.53 for the mildle-class students and 0.38 for the lower-class students. The correlations between ability to conserve and premeasurement concepts were found to be 0.26 for the middle-class students and 0.41 for the lower-class students.

The investigators concluded that "while the correlations between progress in conservation and other measures of mental aptitude and school achievement were only moderately high, they were substantial enough and consistent enough to warrant further investigation."

Overholt (191) explored the relationship between the understanding of the concept of conservation of substance and achievement as measured by a standardized test. He found that the concept of conservation of substance seems to be related to intelligence and to understanding of arithmetic. In a further analysis he found that with adjustments made for differences in intelligence levels between the two groups (conservers-nonconservers), there was no significant difference in mean arithmetic achievement (total score), understanding of arithmetic concepts, and ability to solve arithmetic problems, as measured on the Iowa Tests of Basic Skills, between conservers and nonconservers.

The evidence would suggest to the teacher that there is a relationship between Piaget's tasks and achievement in arithmetic as measured by standardized achievement tests. There is no indication that performance on conservation tasks is a better predictor of arithmetic ability than performance on IQ tests. As Almy points out, "The crucial element may be verbal ability."

How can teachers facilitate the growth of the students through the stages of cognitive development?

Much exploratory work must be carried out before this question can be answered. Piaget's descriptions of typical functioning of children at the various stages of development have implied certain "modes" of instruction that may be more effective than others at specific stages of development. What is needed now is some evidence on what is involved in the transition from stage to stage so that means can be developed for implementing such transitions. As Simon (240) states, "We need to discover how the system could modify its own structure."

Piaget has attempted to interpret this transition from stage to stage in terms of equilibration. Wohlwill (292) points out that this transition is not a "process of equilibration," but rather involves a "disturbance of equilibrium" between internal structure and external assimilations.

Experimenters who have attempted to induce conservation-type behavior in children earlier than it typically would be expected have not met with a great deal of success. Their training procedures can be lumped into three general techniques: direct observation, social reinforcement, or cognitive conflict.

The latter technique appears more in line with Piaget's hypothesis regarding transition by means of "disturbance of equilibrium." Smedlund (244) was able to show significant increase in conservation relative to a control procedure by giving subjects practice in situations involving conflict between expectations based on addition and subtraction and expectations based on changes of form. Gruen (120), comparing direct training procedures with a cognitive conflict procedure, both with and without verbal pre-training, found the cognitive conflict training superior to the direct training procedure. However, he points out that neither training procedure was particularly effective in inducing number conservation. Another interesting finding was that the verbal pre-training alone was about as effective as either direct training or cognitive conflict in the inducement of number conservation. He points out that "an experimenter who uses a verbal test of conservation must be certain that the subjects understand the language he is using. Otherwise, a child capable of conserving may be deemed a non-conserver erroneously."

Wallach and Spratt (280) appeared to have "striking effect" on the development of the concept of conservation by directly teaching 15 "non-conserving" first graders. Their training procedure involved showing the children the reversibility of rearrangements which they, prior to training, regarded as implying changes in number. The concept of conservation attained by the group in the process of training was also retained as indicated by a retention test given about 2-3 weeks later. There was also some transfer to a different conservation task. Wallach and Spratt feel that "experiences with reversibility . . . may be what leads to the development of number conservation, not only in the present experiment, but in normal life."

As stated initially, the teacher must realize that much work must be done before the question of facilitating cognitive development can be answered with any degree of confidence. The hypothesis regarding "cognitive conflict"—that is, the discrepancy between what the child's cognitive structure "expects" and his "real world" assimilations—has received some empirical verification. Also, the concept of "reversibility" may play a crucial role in the development of number conservation.

What are some implications of Piaget's work for the teacher of elementary school mathematics?

One can accept, reject, or ignore the work of Piaget regarding its pertinence to the task of developing curriculum and methods in the area of elementary school mathematics. For example, the writers of the "Cambridge Report" (112) state:

> We made no attempt to take account of recent researches in cognitive psychology. It has been argued by Piaget and others that certain ideas and degrees of abstraction cannot be learned until certain ages. We regard this question as open, partly because there are cognitive psychologists on both sides of it, and partly because the investigations of Piaget, taken at face value, do not justify any conclusion relevant to our task. The point is that Piaget is not a teacher but an observer—he has tried to find out what it is that children understand, at a given age, when they have been taught in conventional ways. . . . If teaching furnishes experiences which few children now have, then in the future such observers as Piaget may observe quite different things.

Irving Adler (1) has stated in a recent article, on the other hand, that "Piaget's theory, properly understood, does have many fruitful implications for the art of teaching." He suggests thirteen such implications which are presented here in outline form.

1. Since the child's mental growth advances through qualitatively distinct stages, these stages should be taken into account when we plan the curriculum.
2. Before introducing a new concept to the child, test him to be sure that he has mastered all the prerequisites for mastering this concept.
3. The preadolescent child makes typical errors of thinking that are characteristic of his stage of mental growth. We should try to understand these errors.
4. We can help the child overcome the errors in his thinking by providing him with experiences that expose them as errors and point the way to the correction of the errors.
5. The child in the pre-operational stage tends to fix his attention on one

variable to the neglect of others. To help him overcome this error, provide him with many situations so that he may explore the influence of two or more variables.

6. A child's thinking is more flexible when it is based on the reversible operations. For this reason we should teach pairs of inverse operations in arithmetic together.

7. The child in the stage of concrete operations has an incomplete grasp of the relations among the subsets of a set.

8. A prerequisite for the stage of formal operations is the ability to carry out combinatorial analysis. All combinatorial analysis is based on the formation of Cartesian products of sets. We can easily teach children systematic ways of forming these products by using tree diagrams and rectangular arrays.

9. Mental growth is encouraged by the experience of seeing things from many different points of view.

10. Physical action is one of the bases of learning. To learn effectively, the child must be a participant in events, not merely a spectator.

11. Since there is a lag between perception and the formation of a mental image, we can reinforce the developing mental image with frequent use of perceptual data.

12. Since mental growth is associated with the discovery of invariants, we should make more frequent use of a systematic search for those features of a situation that remain unchanged under a particular group of transformations.

13. We should be careful not to overdo the formalization of the study of topological relations in the tenth grade and below.

The knowledgeable teacher will notice that many of these implications are not new. A comparison with implications from learning theorists (see, for example, Hilgard [131, pp. 486-87]) will reveal much similarity to what have been stated in the past as principles of good learning for children.

What is the influence of schooling in different cultures on the ability to conserve?

Greenfield (119) studied conservation of liquids in Senegal, the westernmost tip of former French West Africa, where the subjects were children of the Wolof ethnic group. The subjects were divided into nine groups, according to degrees of urbanization, schooling, and age level.

In the separation by schooling, the first group included 49 rural unschooled children; the second, 67 rural children who attended small French-style village schools; and the third, 65 urban school children from Dakar, the cosmopolitan capital of Senegal. Each of these groups was comprised of three age groups: 6-7 years, 8-9 years, 11-13 years. Due to the central control of the Ministry of Education, children attending school received nearly identical educations. Test differences were attributed to differences in urban and rural background.

The experimental situation consisted of a personal interview in the Wolof language with each subject, during which questions concerning conservation were asked. The first part consisted of asking each subject to equalize the water levels in two identical partly-filled beakers; then the experimenter poured the contents of one beaker into a second taller, thinner beaker. The child was asked if the taller beaker contained an amount of water equal to the first or more than the first. In the second part, the experimenter poured the contents of the beaker into six shorter, thinner beakers and the child compared the amount of water in the original beaker with the total contents of the six small ones. The achievement of conservation was said to be present when a child gave equality responses to both quantity comparisons. The data are presented graphically in the chart. (See Greenfield, Chart 1, p. 233.)

There was a wider gap between unschooled and schooled Wolof children than between rural and urban children. By the eleventh or twelfth year virtually all the school children had achieved conservation, but only about half of those not in school had done so.

In addition to compiling the above data, Greenfield studied the

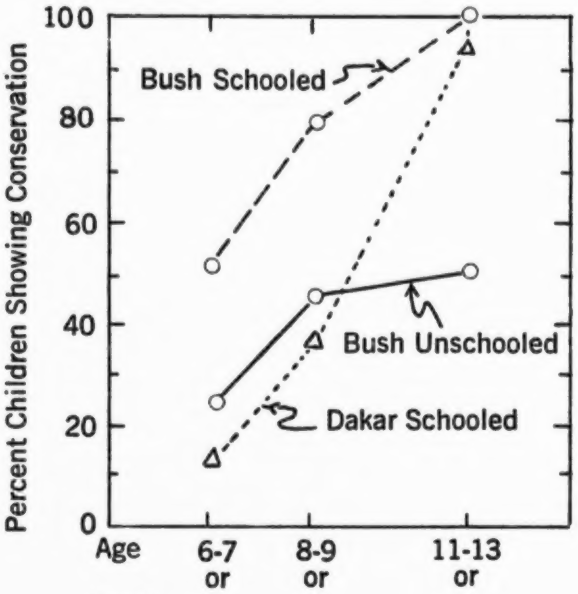

Figure 3. Percent of different backgrounds and ages exhibiting conservation of continuous quantity

justifications the children gave for their answers. These fitted three main classifications: perceptual (features of the display), direct-action (actual pouring), and transformational (internal reasoning). This last class was subdivided again as indirect-action (imagined pouring) and identity (nothing changed). The school children showed early reliance on perceptual reasons followed by a later decline. In contrast, unschooled children showed a gradual rise with age in perceptual reasons.

In most cases Wolof children used transformational or direct-action reasons as a basis for justifying conservation, just as American children do. However, direct-action assumed greater importance for Wolof children. Those giving a transformational reason were generally thinking of identity. Those children demonstrating lack of conservation followed a pattern similar to American children as the majority gave perceptual reasons. In an attempt to hasten conservation, the pouring was performed behind a screen, a technique found successful on American children. This had little effect on Wolof children. The 20 percent minority of Wolof children (primarily unschooled) who gave direct-action reasons seemed to indicate "magical" thinking, attributing special powers to the experimenter who did the pouring. In an attempt to overcome this, another experiment was performed in which the children actually did the pouring. Conservation increased markedly except among the city school children who originally did not have the "magical" thinking.

The experimenter concluded that conservation depended for its development on the presence of a sense of identity, the idea of a potential return to an initial state. It appeared as crucial to conservation in Senegal as in the United States but could be developed by different means.

What do we know about achievement in S.M.S.G. classes?

Among all innovative mathematical programs for elementary school children, the most successful approximation of an integrated program is that of the School Mathematics Study Group (S.M.S.G.). Whereas most innovative programs are the efforts of a single person, aided and abetted perhaps by a small closely-knit group of similarly trained and oriented mentality, the S.M.S.G. program for the elementary grades was planned by a group broadly representative of the community of people interested in children and their proper education. And whereas most innovative programs are concerned with a single mathematical topic or a single approach to one or more topics, the S.M.S.G. program is the product of a sincere effort to develop a mathematically correct, methodologically reasonable, and curricularly integrated program.

Although the projected longitudinal studies of the effectiveness of the program are still in progress, it is possible to make some evaluative statements on the basis of more limited studies completed to date. In reply to a letter to Professor E. G. Begle, Director of the S.M.S.G., seeking a list of significant studies of the elementary grades program, [he] referred the writers to S.M.S.G. *Newsletter No. 15* (230) as "the only significant research" that he knew. Although other studies have been reported in the literature, they are so brief and so lacking in details of design, etc, through no fault of the researchers, that it is virtually impossible to assess their thoroughness or rigor. For this reason the authors of this research monograph confine their remarks to the findings of *Newsletter No. 15*, summary of a report by J. Fred Weaver.

The *Newsletter* summarizes two studies of the Elementary School Mathematics program. One study was carried out to compare actual progress with "expected gains"; the other study was carried out to compare progress of S.M.S.G. classes (the experimental classes) with classes using conventional textbooks (the control classes).

In the first of these two reported studies, data were gathered on the performance of children using the grade four and grade five texts in the fall of 1961 and the spring of 1962. Approximately 600 children were in grade 4 and approximately 1,200 children were in grade five.

As measured by performance on the S.R.A. Primary Mental Abilities Test, the mean IQ for the fourth grade group was 117 and for the fifth grade group 116. The "average" child in these groups was roughly at the 85 percentile. In arithmetic aptitude, the mean for the fourth grade group was 112 and for the fifth grade group 111. As the report states, "Rather clearly, the children in the sample are 'above average' in terms of both index of arithmetic aptitude and the estimated IQ."

Performance in arithmetic was measured by the S.R.A. Arithmetic Achievement Test, Part 1 on Reasoning (Problem Solving) and Part 2 on Computation. A third part on "Concepts" was judged curricularly invalid and the scores were "disregarded for this and other reasons."

The findings are summarized by saying that the mean gains of both grades equaled or exceeded "normally expected gains" in terms of Grade Equivalents. The report asserts that in terms of the test used and its grade norms, the use of the S.M.S.G. sample texts did not inhibit the mean arithmetic achievement of the two groups.

The study went further to assess performance on tests developed to measure content emphasized in the S.M.S.G. texts (number and operation, geometry, and applications). The report states that the performances on these tests "cannot be judged 'good' or 'bad,' or otherwise. The data are normative data and must be interpreted only as such."

Still further, attitudinal (non-cognitive) factors were inventoried through an "Ideas and Preferences Inventory." On six of the nine sub-scales, there was a tendency—"but not a marked one"—in the direction of positive attitudes toward mathematics for both grades. On three of the sub-scales there was a slight tendency in the direction of negative attitudes.

Among the children in grade 5 who had used the S.M.S.G. sample texts in grade 4, the mean fall testing scores were

> . . . indicative of no more favorable attitude toward mathematics than (were) the mean fall testing scores of fourth grade pupils who were using S.M.S.G. sample texts for the first time. But in both grades and for both boys and girls there was a slight tendency for more favorable attitudes at the end of the year than at the beginning of the year.

In the second study reported—an experimental study—the performance of ten experimental fourth grade classes was compared with an equal number of control fourth grade classes using conventional texts. In educational background of the two groups of teachers "there was some indication that the experimental teachers had a slightly better background in mathematics." In salary "the average salary of the experimental teacher was $7,054 compared with $5,490 for the control teacher."

The performance of both groups was measured by the STEP Test 4A in September 1961, and in May 1962. Both groups were administered the D.A.T. (Differential Aptitude Test) in February 1962.

Data gathered permitted Weaver to say that there was no significant difference between the performance of the two groups in the kind of arithmetic measured by STEP Test 4A. The slight difference that was noticeable was in favor of the experimental group. The report concludes that even though there was no significant difference in achievement of the traditional arithmetic measured by the test, the experimental group was exposed to a number of topics that were not in the program of the control classes.

What about the Cuisenaire materials?

The recent increased emphasis on structure in the teaching of elementary school mathematics has brought an increased interest in the materials which claim to use the approach. Among the many materials available, the Cuisenaire materials are among those most widely advertised and discussed. Several studies on the effectiveness of these materials have been reported.

Passy (196) reported a sizeable study using 1,800 New York State third-grade children and controlling for such variables as reading ability, reading levels, mental ability, mental-ability levels, length of child's attendance in the school district, attendance levels, socioeconomic status, teachers' total teaching experience, and years of experience in the school district. His findings are summarized:

1. In computational skills, the Cuisenaire users achieved significantly less computational skill (as measured by the Stanford Achievement Test) than either of the two comparison (non-Cuisenaire users) groups.
2. In mathematical reasoning, the Cuisenaire group(s) achieved significantly less than the non-Cuisenaire group(s) who had a "meaningful" arithmetic program.

Passy concluded, "the data indicate that the third-grade children in this particular program, using Cuisenaire materials, achieved significantly less at the 5 percent level of significance . . . than either of the two samples that were used for purposes of comparison."

Lucow (171) reported a study done in the provincial schools of Manitoba, Canada. He used third-grade children and a criterion test that involved multiplication and division only. He reported that the experimental groups using the Cuisenaire materials had also used these materials in grades one and two and had some instruction in multiplication and division. Hence, these topics "were not altogether new to them" in grade three.

The experimental teaching period in grade three lasted six weeks. Teachers in both the experimental and control classes taught "at an accelerated pace, the content in multiplication and division as authorized for grade three by the Department of Education." Among Lucow's findings are the following:

1. For teaching multiplication and division, the use of the Cuisenaire method is effective. "(But) there is some doubt of its general superiority over current methods of instruction."
2. The Cuisenaire method seemed to operate better with bright and average rural children, but not much better with "dull" rural children. "Urban children thrived just as well under any method at all levels of intelligence."
3. Lucow summarized his opinion about the Cuisenaire method this way: "(it) is a good one and deserves to be included in any grade three teacher's repertoire of methods. On the other hand, the repertoire should not be emptied to allow for only the Cuisenaire method. Children should be taught by whatever method they respond to best. No teacher should limit herself to one method of instruction. . . ."

In both the above studies, and in most studies comparing the effectiveness of materials, the tacit assumption is that any differences found can be attributed to the materials. Brownell (38) raised the more fundamental question: Perhaps the differences, if any, and in whichever direction, are due to the skill and enthusiasm of the teacher, not to the materials. (This phenomenon is generally known as the Hawthorne Effect.)

To test his hypothesis relative to these materials, Brownell studied their use in Scotland and England. He identified the use of these materials as Program A, and the traditional method and materials as Program B. Program A was new in the cooperating Scottish schools. But in England where "the teachers had been exposed to one new system of instruction after another, they tended to view Program A as just another scheme in a long series."

Brownell found that the reason why the children in the cooperating Scottish schools did better was not due to the materials but to the teacher's enthusiasm for the new. Using the new raised her "quality of teaching."

To add further support to his hypothesis, Brownell found that in the cooperating English schools, where the novelty of the material no longer holds, the children in the traditional Program B did better than the children in the experimental Program A.

Brownell states: ("The study) showed that the significant variable . . . was not the two programs . . . but quality of teaching. It bore out the by-no-means-new fact that an instructional program is one thing in the hands of expert, interested teachers, and another thing in the hands of teachers not possessed of these characteristics."

Can children learn the elements of mathematical logic?

Suppes and Binford (261) investigated the teaching of mathematical logic to groups of academically talented fifth- and sixth-grade children. They also extended the study to include the ability of these children to transfer their learning to the reasoning involved in learning the standard school subjects—arithmetic, reading, English, etc.

In the 1961-62 school year, twelve experimental classes of fifth-grade children and constituting approximately the top 25 percent of their peers studied mathematical logic. The 350 children were taught, with one exception, by the regular classroom teachers who had participated in an intensive four-week training session in the prior summer. Most of the classes studied logic three times per week, each lesson being about 30 minutes long.

The content of the instructional program was that of the textbook

Mathematical Logic for Schools by Patrick Suppes and Shirley Hill. The classes completed from 115 to 195 pages.

During the 1962-63 school year most of the prior fifth-grade classes continued the study of the textbook and, including the review work, they completed between 162 and 284 pages of the text. Also, 12 new classes of fifth-grade children studied the program and completed between 117 and 183 pages of the text.

For comparative purposes, two control groups of Stanford University students, with a modal grade between the sophomore and junior years studied logic using the same textbook. One class met in the spring and the other in the fall quarter 1962. The college students completed in four weeks the material that the children completed in one school year.

On the basis of the performance on tests administered to the experimental and control groups, the authors concluded:

> 1. The upper quartile of elementary school students can achieve a significant conceptual and technical mastery of elementary mathematical logic. The level of mastery attained by the children was 85 to 90 percent of that attained by the university students.
>
> 2. The level of achievement can be acquired in an amount of study time comparable to that needed by college students if the study time for the children is distributed over a longer period of time and if they receive considerably greater amounts of direct teacher supervision.
>
> 3. There is anecdotal evidence from teachers which suggests that there is some transfer of the learning in the form of increased critical thinking in such subjects as arithmetic, reading, and English.

Smith (245) reported a critical analysis of the above study, and Suppes (259) presented a reply to Smith's analysis.

How soon should we teach the "basic properties" of the real number system?

Imbedded in this question is the prejudgment that the basic properties should be taught. Rationale for this judgment is stated very succinctly by Bruner (47) when he writes:

> The first object of any act of learning, over and beyond the pleasure it may give, is that it should serve us in the future. . . . (A) way in which earlier learning renders later performance more efficient is through what is conveniently called nonspecific transfer or, more accurately, the transfer of principles and attitudes. In essence, it consists of learning initially not a skill but a general idea, which can then be used as a basis for recognizing subsequent problems as special cases of the idea originally mastered.

In arithmetic the concepts of commutativity, associativity, distribution, identity, and closure, for example, are used over and over again. Each new arithmetic process does not consist of an entirely new set of "rules," but makes repeated use of the basic properties. If introduced formally at too early an age, however, this formalistic-axiomatic study becomes a meaningless and useless endeavor for the child. The problem becomes, as Hartung (125) points out, "At what point in the learning of a particular topic is it proper to use (or encourage the use of) a formal treatment?" He goes on to suggest that for many pupils who have had a modern program in the elementary school this point may be reached in the early junior high school years.

Baumann's study (19) dealt with the performance that could be expected from second- and fourth-grade children on the attainment and use of the concepts "commutativity," "closure," and "identity." Evidence from the study suggested that the attainment of these concepts was quite difficult for the students. Many subjects appeared to approach each task as a new one without considering the experiences they encountered in previous tasks. They did not take advantage of previous instruction. Even the high IQ fourth graders, who did exhibit more success on the task than any of the other subjects, generally indicated a lack of "readiness" for such instruction.

Crawford (67) investigated the age-grade trends in understanding the field axioms. A test of the field axioms was constructed and administered to students in grades 4, 6, 8, 9, 10, and 12. The results indicated that mean scores increased significantly from one even-numbered grade to the next in a manner which was generally linear. No significant differences were found between the scores of boys and girls. Intelligence had an increasing effect on the scores as the grade level increased. The order of difficulty of the axioms from easiest to most difficult was: commutativity, inverse, closure, identity, associativity, and distributivity.

The pieces of evidence cited would seem to suggest that the study of these mathematical structures should not be formal, rigid, abstract, or symbolic at an early grade level—perhaps not in the elementary grades. Certainly children use these basic laws intuitively in much of their early work in arithmetic. As Fehr (90) points out, "Children sense that the sum $3 + 5$ equals the sum $5 + 3$, and that both are 8. To explicitly call this a commutative property at this time of learning is ridiculous." Even at an earlier stage children can see that combining a pile of three pebbles with a pile of four pebbles yields the same result as combining a pile of four pebbles with a pile of three pebbles.

The teacher can feel quite confident that arithmetic organized in such a manner that the "basic laws" emerge as useful structures for learning will be beneficial to the student. The teacher must also be aware of the "operational" level of his students so that appropriate development of the "basic laws" can be carried on meaningfully. A formal, symbolic statement that $a + b = b + a$ may be "ridiculous" at a particular level, but a discussion of why Bill, who took 5 steps along a number line—then 2 more, ends up at the same place as Bob, who took 2 steps—then 5 more, may be entirely appropriate. As Phillips (200) points out, we must be careful not to hinder rather than help the child by starting with the "sophisticated end products" of learning.

What is a desirable arithmetic program for kindergarten students?

In many schools the mathematical development of kindergarten students has been dependent upon incidental mathematical learning opportunities which arise in the everyday experiences of the students. Some confusion may develop over the use of "incidental" in describing the program in mathematics at the kindergarten level. An incidental approach was never meant to be an accidental approach. Informal was never meant to be unplanned. While all children benefit from planned sequential number experiences, the very young are especially dependent upon their teacher for success in number work (32).

Dutton (84) found that during a typical year in kindergarten, without a systematic program, children extend their understanding of numbers markedly. But the study showed that some aspects are neglected, such as the writing of numerals and the solving of simple problems. Stephens (249) compared the time concepts learned by kindergarten children when presented through an incidental approach alone with a method of incidental teaching supplemented by a specific planned program of instruction in these concepts. There was a significant gain in time concepts made by pupils who received specific instruction in these concepts as compared with those pupils who received only incidental teaching of time telling.

Elsewhere in this monograph it has been pointed out that kindergarten entrants come to school with a considerable number of mathematical skills and concepts. These are influenced, of course, by such personal and environmental factors as mental maturity and the socioeconomic level of the family.

Dutton has pointed out that at least one-third of each entering kindergarten class is mature enough and ready for systematic work involving the use of counting, enumerating, grouping, reproducing numerals, and extending other mathematical concepts of size, shape, form, and measurement. He suggests that at the beginning of the year, kindergarten teachers use simple tests to determine their children's needs and abilities. With test findings to guide their work, teachers should be able to pinpoint the skills and understandings that should be taught. Direct teaching may be done either as planned group instruction or as incidental teaching with a sequential plan in mind.

Do the summer educational programs for children of poverty succeed?

Little quantitative evidence is available, but some qualitative evidence may shed some light on the question. A recent report (65) summarizes the conclusions of a group of 27 consultants who visited a sample of 86 school districts in 48 states, including almost all the nation's major cities.

The personal observations of the consultants are summarized as follows:

> 1. The single most widespread achievement of the Title I program is that it is causing teachers and administrators to focus new thinking on ways to overcome educational deprivation. . . . For the most part, however, projects are piecemeal, fragmented efforts at remediation or vaguely directed "enrichment." It is extremely rare to find strategically planned, comprehensive programs for change. . . .
>
> 2. In distinguishing those classrooms that favorably impressed consultants from those that appeared poor, the explanatory factor most frequently observed was the difference in the quality of relationship—the rapport—between teacher and child.
>
> 3. . . . there was frequent lack of involvement of teachers in the formulation of programs they are expected to carry out.
>
> 4. One of the most disappointing findings was the failure of most schools to identify and attract the most seriously disadvantaged children.
>
> 5. Frequently, heavy purchases of educational equipment are made without examining the educational practices that underlie their use.

In summary, the National Advisory Council on the Education of Disadvantaged Children believes that future summer programs, besides being important in themselves, can have special beneficial effects on the year-round success of Title I programs which can be attained in no other way.[1]

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[1] *Summer Education for Children of Poverty.* Report of the National Advisory Council on the Education of Disadvantaged Children. Washington, D. C.: Superintendent of Documents, U. S. Government Printing Office (OE-37006), 1966.

How do children in the United States compare with children in other countries in mathematical learnings?

The most extensive study in comparative education of mathematical learnings is that of the International Project for the Evaluation of Educational Achievement (139) published in 1967 and known as the I.E.A. project. "The overall aim (of the project) is, with the aid of psychometric techniques, to compare outcomes in different educational *systems.* The educational systems studied were those of Australia, Belgium, England, Federal Republic of Germany, Finland, France, Israel, Japan, Netherlands, Scotland, Sweden, and the United States."

Instruments used to gather data included mathematics tests (10 separately prepared test booklets graded in difficulty with a time limit of one hour per test), a school questionnaire completed by the building principal which provided data about certain characteristics of the school, a teacher questionnaire in which the teacher answered questions about her personal background, a student questionnaire, a student opinionnaire which provided data for two descriptive scores and five attitude scores, and a case study questionnaire concerned with the economic development, educational philosophy, etc., of each nation.

Approximately 50 million items of information were collected from the administration of the instruments to 132,775 students, 13,364 teachers, and 5,348 building principals (head masters, or head teachers). The students were measured at two "strategic" school levels—near the end of compulsory schooling and at the end of pre-university schooling. Only the first of these two levels is appropriate to the scope of this research monograph.

Most of the data for this level came from two groups: Group 1a, which consisted of all pupils age 13 years 0 months to 13 years 11 months; and Group 1b, which consisted of all pupils at grade levels where most pupils of age 13-0 to 13-11 were located.

In interpreting the findings it is important to heed the statement of the Council of the I.E.A. project that "The I.E.A. study was not designed to compare countries, . . . it is not to be conceived of as an 'international contest.' . . . its main objective is to test hypotheses which have been advanced within a framework of comparative thinking in education."

The length of the study (two volumes, approx. 700 pages) and the wealth of material cannot be summarized in this monograph. A few typical questions and the answers are:

Is achievement related to the opportunities provided by the school? Opportunities were defined as total hours per week in school, hours per week allocated to teaching mathematics, hours per week of homework, and hours per week devoted to mathematics homework. The study concludes: "for the whole, number of hours per week of schooling seemed to bear little or no relationship to mathematics achievement. . . ."

Is the use of learning-centered or "discovery" approaches as contrasted to drill or rote methods, positively correlated with mathematics performance? To get evidence of the type of method being used, at least according to the pupils' perceptions, each pupil was asked to describe his mathematics class by responding with "agree," or "disagree," or "uncertain" to such statements as:

> My mathematics teacher does not like pupils to ask questions after he has given an explanation.
>
> Much of our classroom work is discussing ideas and problems with the teacher and other pupils.

High scores are taken to indicate the acquisition of such non-cognitive outcomes as promoting inquiry, independent study, and student activity.

It is important for the reader to note that the report uses quotation marks around the word "discovery" to cue in the reader that the term is ambiguous. It is highly unlikely that students replying to this questionnaire had any real understanding of or exposure to a method of teaching that approximated *true* discovery procedures. (See the section on discovery in this monograph.)

Nevertheless, the conclusion on this question is that "a positive relationship between 'discovery' approaches and interest in mathematics existed at the 13-year-old level."

Do boys excel girls in mathematics achievement at the 13-year-old level? It was hypothesized that girls would excel boys in problem solving, and that boys would excel girls in computational skills. Of the 42 comparisons made, all but 2 came out in favor of the boys. The mean sex difference in Group 1a was highest in Belgium, Japan, and the Netherlands, and also in Group 1b, in these three countries plus England. The smallest differences were found in the United States and Sweden, where the boys only slightly excelled the girls at the 13-year-old level.

Do children in the United States have as much opportunity to learn the topics as children in other countries? Data on this question were gathered through a questionnaire. In both Groups 1a and 1b, children of the United States had next to the lowest opportunity to learn the topics covered in the examinations.

Several additional studies ranging over a quarter of a century have been done to compare achievement in general and arithmetic in particular of British and American children. One of the first attempts to put comparative education on a scientific basis was the use of American achievement tests in Fife, Scotland, in 1934. In modest terms the Scottish Council for Research in Education (267) says: "The results were not unfavourable to the primary schools in Scotland."

Elsewhere in the same volume it is reported that the administration of a battery of American achievement tests to a complete age-group, "eleven-year-olds," in all Fife schools, revealed that, in general level of achievement, Fife "eleven-year-olds" are 16 months ahead of American children of the same age. A subsidiary investigation substantiated that the lead held at 11½ years is also present at 7½ years. This advantage of Scottish pupils in education achievement is attributed mainly to the extra 15 months of schooling. "It would thus appear that the practice customary in Scotland of admitting children to school a year or fifteen months earlier than in America is justified on educational grounds."

A quarter of a century later (1958) Buswell (52) reported the results of the administration of an English-made test to approximately 3,000 English children and an adapted form of the same test to approximately 3,000 children in central California. The test was administered to children in the age-range of 10 years 8 months to 11 years 7 months as of the month in which the test was given. Both groups included approximately equal ratios of rural and urban children. "In England the test was given to schools selected by the technique of 'random numbers sampling' and the same method was followed here."

Of the 100 items on the original test, 70 were free of cultural bias. Achievement of the two groups of children was compared on performance on the 70 items. The test was divided into two sections—computation and problems.

The mean scores on the total test were 29.1 for the English children and 12.1 for the California children. The difference between the scores is statistically significant at well beyond the 1 percent level.

In a study comparing arithmetic achievement of American and Dutch children, Kramer (164) found Dutch children in Grades 5 and 6 to be significantly superior to American children on tests of arithmetic problem solving and arithmetic concepts.

How do New York State and English children compare in their knowledge of basic mathematical understandings? Most available comparative studies use tests of computational skills, concepts, and verbal problem-solving ability. Any attempt to gather evidence of understandings is indirect. Pace (193) sought to answer the question above by using a modified form of Glennon's *Test of Basic Mathematical Under-*

standings. This was administered to a random sample 2,692 English pupils (in 60 schools) in their sixth year of elementary education, to 1,616 fifth-grade pupils, and 1,590 sixth-grade pupils in central New York State.

Pace made three comparisons in order to study the effect of the English child's entering school (and systematic instruction in arithmetic) at age five while the New York State children begin at age six:

1. For pupils of the same age, but differing in the number of years of formal instruction in arithmetic,
2. For pupils with the same number of years of formal instruction in arithmetic, but differing in age, and
3. For pupils of the same age and with the same number of years of formal instruction in arithmetic.

There was a significant difference in favor of the English children with six years of instruction over the New York State children with five years of instruction.

There was a significant difference in favor of the New York State children who had six years of instruction over the English children who also had six years of instruction.

When age range was held comparable and both groups had the same number of years of instruction, "there was no statistically significant difference between the two groups" in their knowledge of basic arithmetical understandings.

In other studies (using the Buswell test, referred to above) by Bogut (29), Thomason and Perrodin (269), Tracy (273), the general findings are that English children achieve higher scores than subpopulations of United States children.

Are European textbooks more rigorous than United States' textbooks? Among others, a reason often cited for alleged superiority in mathematical achievement by European children is that the textbooks used in these countries contain more mathematics than do textbooks in the United States. Dominy (79) made an analysis of the content of representative sets of textbooks used in England, France, West Germany, the Soviet Republic, and the United States. Comparing books used up to age eleven and using thirteen commonly taught topics, she found that the United States textbooks had a lighter content load than those books commonly used in England and France, and a heavier content load than those commonly used in West Germany and the Soviet Republic. Dominy concluded that if any alleged superiority of children in these four European countries does in fact exist, it is not possible to say that it is due mainly to more intensive or rigorous textbook programs.

Part Two

Studies Concerning the Child

Can children learn anything that adults can —and more efficiently?

This question did not appear in the prior editions of this research monograph because professional workers generally assumed that children could *not* learn anything that adults could learn—and more efficiently. The reason for the question appearing in this edition is due in large part to an oft-quoted statement since 1960. In *The Process of Education* (47), which is a summary of the ideas of many persons in attendance at the Woods Hole Conference, is the statement "any subject can be taught effectively in some intellectually honest form to any child at any stage of development."

Another statement from the same book makes a still stronger assertion:

> In teaching from kindergarten to graduate school, I have been amazed at the intellectual similarity of human beings at all ages, although children are perhaps more spontaneous, creative, and energetic than adults. As far as I am concerned, young children can learn almost anything faster than adults can if it can be given to them in terms they can understand.

The sweeping nature of the two statements has brought replies from several scholars. Space permits only two—one from a mathematician, the other from an educational psychologist with specific competence in cognitive processes.

Morris Kline (162), a mathematician, discusses the first statement as follows:

> The problem of curriculum reform has been confounded still further by other professors. Jerome Bruner and other psychologists—no doubt intending no more than to teach what has been gathered from recent investigations on the learning capacities of young children—have, however, affirmed broadly

that there is no idea, no matter how abstract, that cannot be taught in some intellectually honest form to a child at any stage of development. The saving feature of this doctrine is its vagueness. Leaving aside the question of whether the particular abstractions that any one group may be interested in promoting warrant priority, one wonders how one could present the substance of Kant's "Critique of Pure Reason" even to high school students. What can happen and does happen when the doctrine is taken too seriously so that students accept the abstraction docilely and are as understanding and as critical of what they are taught as children are when they learn a catechism.

David Ausubel (15), a scholar in cognitive processes, discusses the accuracy of the statements:

> Although both propositions are generally untrue and unsupportable, they are nevertheless valid in a very limited sense of the term.
>
> Generally speaking . . . adolescents and adults have a tremendous advantage in learning any new subject matter—even if they are as unsophisticated as young children in the subject matter to be learned. For older learners are able to draw on transferable elements of their overall ability to learn concepts abstractly. Hence, they are able to move through the concrete, intuitive phase of intellectual functioning very rapidly.

What do we know about readiness for arithmetic learning?

At the time of the second revision of this monograph, two commonly held misconceptions of readiness for learning arithmetic were discussed. The first was that readiness is an "either-or" or dichotomous state. The second was that readiness is something with which the primary grade teacher alone needs to be concerned. Any contemporary discussion of readiness would have to include, it would seem, another topic—the interpretation given to Bruner's (47) oft-quoted statement, "Any subject can be taught effectively in some intellectually honest form to any child at any stage of development."

Swenson (265) discusses the first misconception by saying that:

> Only to the uninitiated can readiness appear to be a simple matter of reaching some mythical, magical point preceding which the learner is clearly not ready and following which he is clearly and unequivocally ready to learn. . . . Adults will understand the learning of children much better if they will think in terms of their being more ready or less ready rather than ready or unready.

That the problem of readiness is not alone the concern of the primary grade teacher is evident when one considers the highly systematic structure of arithmetic. Brownell (35), in his work with fifth graders, found that children often experience difficulty with long division because they have not acquired an adequate mastery of the skills and facts prerequisite to successful learning of that process. The more recent work of Gagne *et al.* (102) has indicated the importance of order of acquiring subordinate knowledges in a knowledge hierarchy as an important factor in mathematics learning.

The studies of Brownell and Gagne point to the importance of subject-matter readiness. Callahan (57) has recently pointed to the multidimensional nature of readiness for learning elementary school mathematics. For example, the willingness or acquiescence of the pupil to learn (affective readiness) may play an important role in optimal learning. The interrelatedness of "subject-matter" readiness and "affective" readiness is indicated in Slavina's (242) research cited in another section of the monograph. The "intellectual passiveness" was successfully overcome (affectively the children were ready), but it was found that they lacked the number skills essential to the mathematical learnings being attempted (subject-matter readiness).

Bruner's statement cited earlier in this section needs some clarification. Although Kline (162) states that, "The saving feature of this doctrine is its vagueness," this very feature may also have contributed to the "incalculable mischief" that the generalization of the statement has wrought in an entire generation of overeager and uncritical curriculum reform workers (14). Some reformers may see it as license to put the most abstract and formal mathematics at primary grade levels.

It seems that the statement must be interpreted in the light of a "stage" theory of cognitive development. Piaget's stages have already been discussed in another section. Bruner (48) talks in terms of three stages: (a) Enactive, (b) Iconic, (c) Symbolic. Regarding mental development, Whitehead (285) also discusses three stages: (a) Romance, (b) Precision, (c) Generalization. Whatever the labels attached to the stages, the important point would appear to be that children undergo changes in their cognitive capacity from infancy through adolescence and that the functioning at a particular stage is necessary and builds readiness for efficient functioning at the next successive stage. The four-year-old who says, "I'm 4 and Marty is 2, and when I'm 5 Marty will be 3, and when I'm 6 Marty will be 4," while clearly giving evidence of a primitive, intuitive knowledge of the function idea, does not possess the perceptual-motor, emotional, or cognitive readiness to attempt a study of functions; but the experience: ". . . holds within itself unexplored connection with possibilities half-disclosed by glimpses and half-concealed by the wealth of material" (285).

The elementary school mathematics teacher can feel very confident that a readiness program at all grade levels will facilitate subsequent learning. This would include readiness built on the acquisition of subordinate knowledge as well as readiness of the student to develop continually higher levels of cognitive and affective functioning in the acquisition of the substantive matter of elementary school mathematics.

What mathematical concepts are possessed by the child when he enters school?

Two and three decades ago this question was important because of decisions on whether to have systematic instruction in arithmetic in the first and second grades of the schools. As a result, considerable research was carried out to ascertain the mathemetical concepts of 6- and 7-year-old children at time of entrance into first grade. Presently the problem has moved down to the entering kindergarten child and the trend is toward earlier schooling, especially for children from specific segments of our society.

Brownell (34) compiled and analyzed the early research on this question. Following are some selected findings from his summary. He concluded that the following skills and concepts seem to be quite well developed by the time most children start school: rote counting by 1's through 20; enumeration through 20; identification of number through 10; with objects, the concepts "longest," "middle," "most," "shortest," "smallest," "tallest," "widest"; exact comparison or matching through 5; number combinations with objects to sums of 10; in verbal problems adding 1 and 2, and probably most facts with sums to 6 or 7; unit fractions through halves and fourths as applied to single objects; ordinals through "sixth"; geometric figures "circle" and "square"; telling time to the hour; recognition of all times to the half hour. Extensions of these skills and concepts as well as others were developed to quite a high degree by significant numbers of students in the various samples. The teacher interested in a further breakdown of these concepts and skills should go directly to Brownell's work.

Presently, studies have focused on the skills and concepts possessed by the 5-year-old when entering kindergarten.

Sussman's (264) findings give some evidence that today's kindergarteners know as much about arithmetic at the beginning of kindergarten as first-grade children did a few decades ago. Studies by Williams (286) and Bjonerod (26) tend to substantiate this statement. Some selected findings by Williams and Bjonerod would indicate that of the present kindergarten entrants 50 percent or more can: rote count to 19; identify one to nine objects; locate the "first," "second," or "fourth" object in a series; select quantities of 3 or less from a larger group; respond accurately to situations requiring an understanding of "largest," "smallest," "tallest," "longest," "most," "inside," "beside," "closest," "farthest"; read the numeral 4; successfully solve number story problems requiring addition and subtraction with sums less than 5; recognize a clock and calendar as instruments used in measuring time; recognize one-half and one-third of one item. About 25 percent to 50 percent can: reproduce 7, 9, 14 by marking the corresponding number of objects; read the numerals 6, 9, and 0; recognize the yardstick (ruler), scale, thermometer as instruments used in measuring; recognize time on the full hour when referring to a clock; realize there are 5 pennies in a nickel. Extensions of these skills and concepts, as well as others, were developed to quite a high degree by significant proportions of students in the two samples. The teacher interested in further breakdown of these concepts and skills should go directly to the sources cited.

Dutton (84) administered the Metropolitan Readiness Test to 236 kindergarten entrants. He concluded from his results that kindergarten pupils come to school with wide and varied backgrounds in number experiences. At least one-third of each entering class is mature enough and ready for systematic work involving the use of counting, enumerating, grouping, reproducing numerals, and extending other mathemetical concepts of size, shape, form, and measurement.

Brace and Nelson (30) attempted to determine the child's understanding of number concepts as revealed by his manipulations of objects rather than by his verbalizations, since there appeared to be some difference between what the child says he knows and how much he knows about what he says. Using Piagetian-type tasks the following selected conclusions and implications were suggested:

> 1. The preschool child's ability to count is not a reliable criterion of the extent to which he has developed the true concept of number.
>
> 2. Since four-fifths of the children had no knowledge of the invariance of number and tended to believe that the number of objects in a group changed when the arrangement was disturbed, it seems safe to conclude that preschool children have a very limited knowledge of the nature of cardinal number.
>
> 3. Since the concept of ordinal number contributed most to the total common variance within the sample tested, and was the biggest contributor to differences, wherever significant differences were found, and, further, since the relationship of this concept to counting was found to increase with age while that of cardinal number to counting decreased with age, it seems safe to conclude that the concepts of ordinal number and cardinal number do not develop concurrently as is generally believed.

4. A thorough understanding of cardinal number, ordinal number, and rational counting must be established before children are able to understand the concept of place value.

5. The sex of the child does not appear to be a factor in the early development of the concept of number.

6. Since children from homes of high socioeconomic level were significantly superior to those from homes of lower socioeconomic level in their number knowledge, it would appear that environmental factors are important in the child's development of the concept of number.

Williams (286) ran correlations between certain psychological and sociological factors and achievement in mathematical concepts at the time of entrance to kindergarten. He found a significant relationship existed between mathematical skills and concepts possessed at time of entrance to kindergarten and: mental maturity; socioeconomic status; rote counting ability; playmate status; frequency of playing counting games with spinners, dice, and cards; knowledge of age, house number, telephone dialing, TV channel numbers, and songs involving numbers.

Kindergarten teachers can feel quite confident that the entering child has accumulated a considerable reservoir of skills and concepts in mathematics. They must be aware, however, of the distinction between verbalizations of concepts and performance which demonstrates the stability of the concepts. Also, evidence would suggest that there is a great deal of variability within a group of typical kindergarten entrants.

Does the age at which a child enters first grade have an effect on subsequent achievement in elementary school mathematics?

A cluster of studies (16, 61, 62, 77, 116, 140), all rather similar in design have been carried out which give evidence pertaining to this question. In general, children who were less than six years of age when entering first grade were compared, at later grade levels, with children who were over six years of age when entering first grade. Where kindergarten experience, sex, and IQ are controlled, the findings generally support the following conclusions:

1. The chronologically older child appears to have the advantage in arithmetic achievement (as measured by standardized tests) over the younger child when given the same school experience.

2. In general, chronological age has more effect on the academic achievement of boys than on that of girls. In one case (16), it was

reported that the differences between boys and girls in achievement were greater than the differences between overage and underage children.

The teacher can feel quite confident that chronological age on entering the first grade is a factor in successful achievement when the school makes no accommodation for this age factor. It would further appear that this factor is more important in boys than girls. From the evidence it would seem that a child, especially a boy, with an IQ of about 100 or less has an increasingly better chance of achieving average progress through the grades the older he is when he enters school.

How do children develop an abstract concept of number?

A study of the developmental process by which the young child arrives at an abstract concept of number was performed by Wohlwill (291) on 72 children, ranging in age from 4:0 to 7:0. The children were individually given a series of matching-from-sample tests of varying difficulty in which each child was presented a stimulus card designating a particular number and required to choose from among the choices the one depicting the same number as the sample. As a training series, sample cards, featuring 2, 3, or 4 dots in varying configurations, none of which was identical to any of the choice cards, were presented one at a time to the child for matching. Five children out of 77 who were unable to give six consecutive correct answers within 48 trials were discarded from the study.

The test series consisted of 7 tests, listed here in increasing order of difficulty, as hypothesized by the experimenter. (See Wohlwill, Figure 1, p. 349.): (A) *Abstraction.* The choice and sample cards varied in number (2-4), form (square, circle, and triangle), and color (green, red, blue) in such a way that any given sample card matched each choice card on only one of the 3 dimensions. (B) *Elimination of perceptual cues.* The choice cards were those of the training series while the sample cards contained rectangles drawn in outline and divided into 2, 3, or 4 equal adjacent squares. (C) *Memory.* The subject matched the training stimulus card to the position of the corresponding choice stimulus of the training series when the latter was removed from view. (D) *Extension.* The choice cards, as well as the sample cards, contained 6, 7, or 8 dots in varying configurations. (E) *Conservation of number.* The child correctly matched the number of buttons with the choice cards of Test D, assisted if necessary by the examiner. The examiner then scrambled the configuration of the sample while the subject watched. The subject was

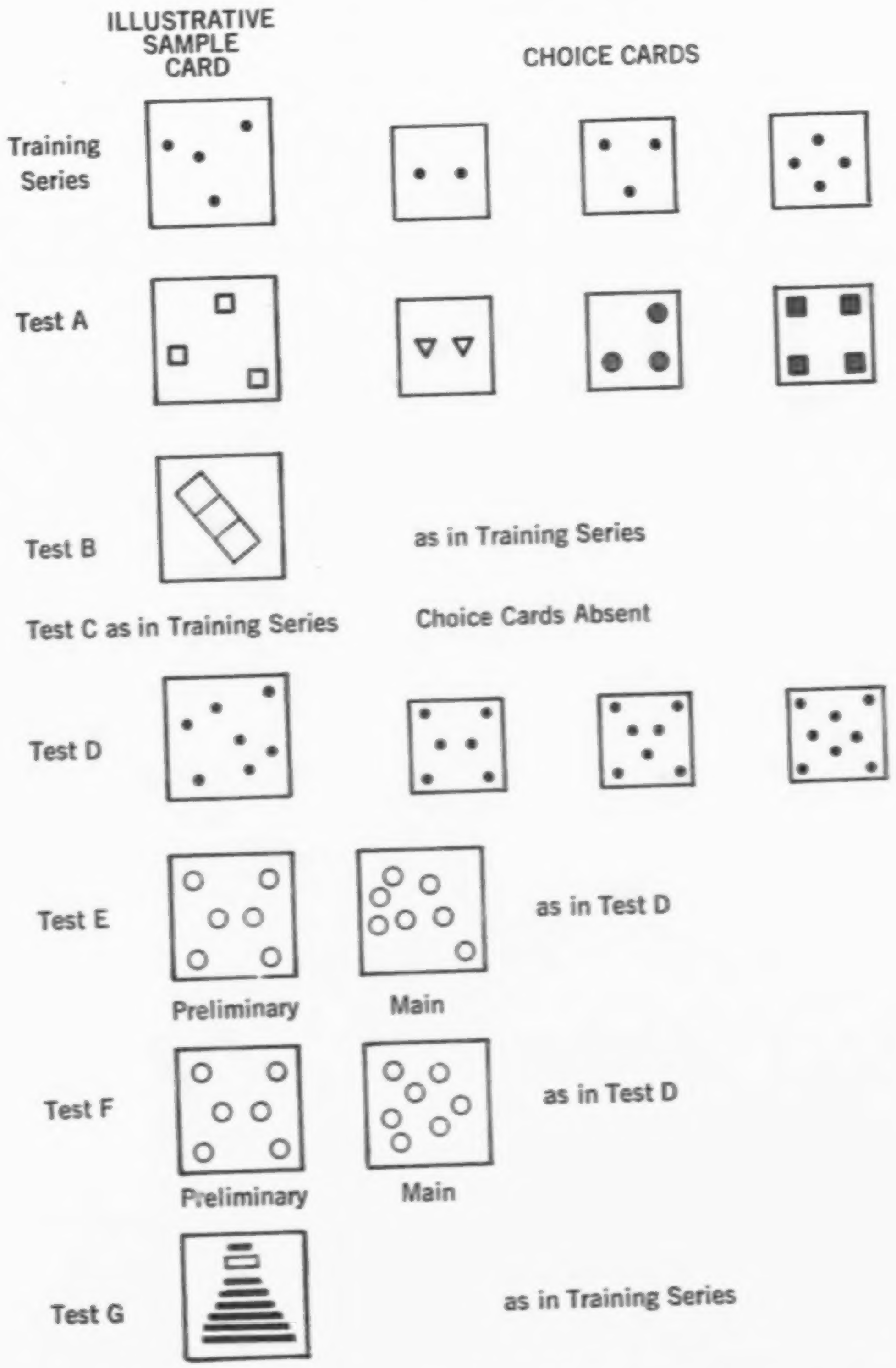
ILLUSTRATIVE SAMPLE CARD
CHOICE CARDS
Training Series
Test A
Test B
as in Training Series
Test C as in Training Series
Choice Cards Absent
Test D
Test E
Preliminary
Main
as in Test D
Test F
Preliminary
Main
as in Test D
Test G
as in Training Series

Figure 1. Sample and choice cards used in training and test series

asked to match the rearranged sample with the choice cards. (F) *Addition and subtraction.* Test F differed in that while the child watched, a button was added or subtracted from the collection immediately following the configurational match, and the subject was asked to match the new set with the correct choice card. (G) *Ordinal-cardinal correspondence.* The subject was asked to match a sample card containing 8 solid bars of increasing length, one of which was colored red to signify the cue-bar, with the training series choice cards. Two supplementary tests, not included in the data analysis, were also given. They consisted of a second number-conservation test involving sets of blocks and buttons and a counting exercise from 1 to 10, both forward and backward.

While the order of difficulty is as listed above, the tests were administered in the order C, A, B, G, D, E, F, with E and F mingled in 12 trials. Except for D, which was presented for 12 trials, the tests consisted of 6 reinforced trials each, and 5 correct responses out of 6 was the criterion for passing. Figure 2 shows the test results. (See Wohlwill, Table 1, p. 356.) These results show only two discrepancies from the hypothesized order: (a) Test B was passed by slightly more children than Test A. It was assumed that the distraction of color and form was more powerful here than the concept of number. (b) Subjects scored higher on Test F than on Tests D and E. Wohlwill attributed the increased difficulty of Test D to the fact that the higher number concepts no longer could benefit from direct perceptual support, thus necessitating symbolic counting. Test E proved more difficult than Test F, as the addition-subtraction understanding seemed to be a prerequisite for success on the conservation tasks.

B.	Elim. of Percep. Cues	49
A.	Abstraction	46
C.	Memory	32
F.	Addition and Subtraction	24
D.	Extension	21
E.	Conservation	14
G.	Ordinal-Cardinal Corresp.	6

Figure 2. Number of subjects passing each item

Wohlwill further analyzed the data by means of a scalogram, a technique for determining whether a sequence of tasks is such that the mastery of any given item presupposes, in general, success on all easier items. Although the scalogram rating for these tests was satisfactory, the experimenter felt that the tests did not represent a series of fixed and equally distinctive steps on the developmental scale. Rather, he suggested that the developmental process could be more adequately described in three fairly discrete phases: (a) number is responded to wholly on a perceptual basis; (b) perceptual support is reduced as mediating structures, that is, the internalized symbols representing the numbers, are developed; (c) the relationship among the individual numbers is conceptualized, leading to such understandings as conservation and cardinality-ordinality.

What are some characteristics of mathematically gifted students?

Characteristics of mathematically gifted students have been compiled by various writers (133, 149, 293). The following list of intellectual characteristics by Weaver and Brawley (283) may be considered as typical.

> 1. Sensitivity to, awareness of, and curiosity regarding quantity and the quantitative aspects of things within the environment.
>
> 2. Quickness in perceiving, comprehending, understanding, and dealing effectively with quantity and the quantitative aspects of things within the environment.
>
> 3. Ability to think and work abstractly and symbolically when dealing with quantity and quantitative ideas.
>
> 4. Ability to communicate quantitative ideas effectively to others, both orally and in writing; and to readily receive and assimilate quantitative ideas in the same way.
>
> 5. Ability to perceive mathematical patterns, structures, relationships, and interrelationships.
>
> 6. Ability to think and perform in quantitative situations in a flexible rather than in a stereotyped manner.
>
> 7. Ability to think and reason analytically and deductively; ability to think and reason inductively and to generalize.
>
> 8. Ability to transfer learning to new or novel "untaught" quantitative situations.
>
> 9. Ability to apply mathematical learning to social situations, to other curriculum areas, and the like.
>
> 10. Ability to remember and retain that which has been learned.

Haggard (122), in his longitudinal study of 45 highly gifted children, made comparisons among high achievers in reading, spelling, language, and arithmetic. His findings shed light on some non-intellective factors characteristic of gifted achievers in mathematics. He writes:

> The high achievers in arithmetic, those who did better on the arithmetic test than would be expected in terms of their over-all level of achievement, tended to see their environment as being neither threatening nor overwhelming. Rather, they viewed it with curiosity and felt capable of mastering any problems they might encounter. In viewing their parents and other authority figures, and in their relations with them, they showed less strain than the high general achievers and the high achievers in reading, and greater independence than the high spelling achievers. Furthermore, the arithmetic achievers had by far the best-developed and the healthiest egos, both in relation to their own emotions and mental processes and in their greater maturity in dealing with the outside world of people and things.
>
> The high arithmetic achievers could express their feelings freely and without anxiety or guilt; were emotionally controlled and flexible; and were capable of integrating their emotions, thoughts, and actions. Similarly, their intellectual processes tended to be spontaneous, flexible, assertive, and creative. Of the subgroups studied, the arithmetic achievers showed the most independence of thought, were best at maintaining contact with reality and at avoiding being bound by its constraints, and could function most effectively in the realm of abstract symbols.
>
> In their relations with authority figures and peers, they were more assertive, independent, and self-confident than were the children in the other subgroups. Generally speaking, they related well to others, but, if they felt that attempts were being made to impose undue restrictions upon them, they tended to respond with hostility and self-assertion in order to maintain their independence and autonomy of thought and action . . . .
>
> The high achievers in arithmetic showed a cluster of personality and intellectual characteristics which are considered extremely desirable in our society. These include a healthy ego, which is relatively free from conflicts and anxieties; ability to act independently and to get along well with others; and such intellectual qualities as creativity, flexibility, and the ability to deal handily with abstract symbols and relationships.

Kennedy and Walsh (156) in a factor-analytic study of giftedness between 90 mathematically gifted high school students and 63 high school students in the same general ability range found:

> A consistent trend for high ability in mathematics to be related to factors on personality tests which can best be described in terms of being aware of the power structure, concerned with theoretical issues as opposed to social issues, and to show some signs of what we have called emotionality, but which may well relate to a typical unconventional reasoning and divergent thinking. . . . As far as the intellectual variables are concerned, there is evidence of a strong factor relating to high achievement and high ability which would seem to indicate that mathematical ability is not a specific ability, but relates to overall high ability. . . .

Glennon (109) gives a warning to teachers in their use of characteristics and factors ascertained from tests for purposes of identifying the gifted students in mathematics when he writes:

> Tests. . .tend to be more oriented to the life of the middle and upper class child than to the life of the lower class child. To the degree the tests are thus oriented, they tend to discriminate against the child from the lower socioeconomic class. Hence, the teacher needs to use extra care to make sure that he does not exclude the child who is talented but whose measured intelligence and achievement scores do not clearly indicate his talent.

Another point regarding the various psychological characteristics of a "gifted" child is that these characteristics are not uniformly displayed by every gifted child. No teacher should expect that because a particular child has been described as "gifted" he will engage in all those behaviors ascribed to the gifted child. However, most gifted children possess the capabilities and potential needed for such display.

What mathematics should be provided for the mathematically gifted child?

Two general approaches are available to the teacher desirous of providing appropriate material for the mathemetically gifted student—acceleration and enrichment (109). Academic acceleration focuses on the characteristic of the gifted student that suggests he can do whatever the average student can do—and do it faster. Thus, acceleration allows the gifted student to travel through the mathematics that has been judged desirable for the average elementary school student at an increased rate.

Enrichment focuses on other characteristics of the gifted student such as his ability to see relationships, patterns, and structures of mathematical systems. Also, as Gallagher (103) points out, enrichment would refer to those activities that stimulate productive and evaluative thinking. Thus, enrichment allows the gifted student to broaden and deepen his mathematical insight by introducing new but related topics as well as deepening insights into what is presently taught to the average elementary school student. These two general approaches need not be mutually exclusive, but can be interrelated to facilitate the overall development of the gifted student. Administrative accommodations for handling these approaches are discussed in a separate section of this monograph.

Jacobs *et al.* (143) reported on a program for the mathematically talented elementary school students, grades 3-6, which was primarily of an accelerated nature, but also included some enrichment. One experimental group started at the third-grade level. A student in that group who could finish the arithmetic of the third grade could then proceed to the fourth-grade book in *another* basal series. Another group began the acceleration at the fourth grade. The student would finish the arithmetic of the fourth grade and then proceed to a fifth-grade book in *another* basal series, etc. Still another group began at a fifth-grade level, and the remaining group had only the sixth grade (one year) of acceleration. Students had the opportunity to complete 5 years of mathematics (books 3-7) in four years. One aspect of the evaluation attempted to compare the gain of students who began the accelerated program in grade 3 and continued in it through the end of grade 6 with groups that began at the fourth grade (in program 3 years), fifth grade (in program 2 years), and sixth grade (in program 1 year). When intelligence was controlled, the only group means which were significantly different were those of the groups who had started at either the third or fourth grade. This group scored higher than the group which had just started at the beginning of the sixth grade. This was true only for the "concept" part of the test. There were no differences in problem-solving ability as measured by the standardized achievement test.

Mullins (186) prepared materials of an enrichment nature for 16 sixth-grade classes in three different communities. The materials were designed to challenge the upper 20 percent of the classes, and were in a form that made it easy for the student to use. Teachers in the study were given a guide, but could use the materials as they saw fit. Mean differences between matched pairs of students on an arithmetic problems test, designed for the study, were significant in favor of the group using the enrichment materials in one community, but not in the other two.

Suppes (262, 258) has reported on research he is carrying out with 40 gifted first graders. In this study, initiated in December 1963, students were allowed to progress through work in the books, *Sets and Numbers* and *Geometry for the Grades,* at their own rate. During the following summer the majority of these students also participated in small group work that explored topics in plane and solid geometry, logic, and the isomorphism of 2 x 2 matrices. Early results indicated that the average group completed approximately 1¾ years of the curriculum in the 26 week period. An interesting finding points up the variation in such a homogeneous (IQ's of 120+) group. The "fastest" student in the group of 40 was approximately 1½ years ahead of the "slowest" student at the end of the 26 week period. The "fastest" student also had a consistently lower error percentage in comparison with the least proficient.

The illustrative research studies cited should suggest that gifted students can handle more complex and abstract mathematics and also can learn faster. Even within homogeneously grouped gifted children there will be much variability. Accommodation to the speed, breadth, and depth of ability of the gifted is aided by the ability of many of the gifted children to work independently. As Gallagher (103) states:

> The dimension of *independence* appears to be a particularly differentiating feature of gifted children and this fact has some obvious implication for educational planning.

What are some characteristics of the educable mentally retarded child in arithmetic?

Burns (49) has analyzed research dealing with this question:

1. The educable mentally retarded students are retarded in their knowledge of arithmetic vocabulary.
2. The educable mentally retarded students are inferior to normal students in ability to solve abstract verbal problems.
3. The educable mentally retarded students are better at solving concrete problems than abstract problems.
4. The educable mentally retarded students have less understanding of the processes to be used in a problem situation and are more apt to guess at the process than normal students.
5. The educable mentally retarded students are more careless than normal students in their work, use more immature processes, and make more technical errors.
6. The educable mentally retarded students are less successful in differentiating extraneous materials from needed arithmetical facts than normal children.
7. The educable mentally retarded students can do equally well with word problems and mechanical operations if the instruction has been meaningful to them.
8. Educable mentally retarded students have little concept of sequence or time.
9. Educable mentally retarded students do better with addition and subtraction; need more emphasis on multiplication and division.
10. Arithmetic readiness is even more important to the educable mentally retarded than the normal students.

What mathematics should we teach the mentally retarded, and how should we teach them?

Burns (49) suggests that retarded children with a chronological age of 6 to 8 years should not have the experiences of the usual second- or third-grade groups, and that the work should more likely be in the nature of a readiness program normally found in kindergarten and first grade. For the educable mentally retarded adolescent, much of the work in arithmetic likely should be in connection with the program of occupational education. Kirk and Johnson (160) state that these children, in general, tend to achieve between the third and fifth grade in their arithmetic abilities, and some are unable to achieve even third-grade ability. They further state that the context should be carefully chosen on the basis of two principles: (a) the content must include the knowledge, skills, and concepts that will be of more value to them now and in later life, and (b) the methods used should be determined by the special disabilities or abilities of mentally handicapped children.

Costello (66) compared the effectiveness of three methods used in teaching arithmetic to mentally handicapped children. The methods were: (a) socialization, an active, experiencing method; (b) sensorization, a method in which concreteness of presentation was used; and (c) verbalization, a method in which verbal description or telling is used. Costello found socialization to be the most effective method, followed closely by sensorization.

Finley (95) presented 20 arithmetic problems in three different contexts to a group of 54 normal third graders and 54 educable retarded subjects in upper elementary and junior high school special classes. The three contexts were: (a) concrete—an instrument individually administered and in which money and actual objects were used to illustrate the problem; (b) pictorial—an instrument group-administered where the items were illustrated by drawings and pictures; (c) symbolic—a group-administered testing instrument with no illustrations or objects (computational). The same computational combinations were used in each instrument and were equally divided among the 4 basic processes. The retarded group performed best on the symbolic (computational) test and worst on the concrete test. Because the concrete test had been individually administered, the experimenter wondered about the possibility that a child feels less conspicuous and hence more at ease and able to achieve better in a group situation than when he is singled out. It should be noted that lack of control of such important experimental variables as teacher variable and the effects of practice may have caused complications that would make the findings questionable.

Bradley (31) working with 15 mentally retarded youngsters in a residential center attempted to determine whether a commercial program on "telling time," designed for normal students, could be used with retarded subjects. Her findings suggested that mentally retarded subjects can profit from a teaching machine program written for normal children.

The primary advantage seemed to be the rapid determination of problems of the children involved in learning the task. An interesting observation was that the subjects seemed to require additional reinforcement to that given by the machine. The majority of the subjects looked at the teacher for approval after completion of each frame. It appeared as though the examiner was essential to the learning situation for the purpose of encouraging sustained attention to the task.

The teacher of the mentally handicapped child can feel confident that an arithmetic program is desirable if it makes use of socially significant situations and includes a direct and systematic program of instruction with a minimum dependence on transfer of training for the learning of new skills. It would appear that the retarded child can learn from a highly structured procedure in the classroom and, where the gifted child can be characterized as "independent" in his study skills, the mentally retarded student will tend toward the "dependent" end of this continuum.

Is achievement in elementary school mathematics affected by "cultural deprivation"?

The tendency has been to describe the achievement of culturally deprived children in terms of their deviance from the norms of children from the homes of middle class parents (115). In general, this deviance from the norms increases as the culturally deprived children progress through the grades. Deutsch (76) has labeled this the "cumulative deficit phenomenon." Writing in regard to this phenomenon between the first and fifth grades he remarks:

> Essentially, it would appear that when one adds 4 years of school experience to a poor environment, plus minority group status, what emerges are children who are apparently less capable of handling standard intellectual and linguistic tasks.

Callahan (58) reported large deviations between expected arithmetic achievement and actual arithmetic achievement of bright eighth grade students in a school whose students were culturally deprived.

Montague (183) studied the effect of socioeconomic background on arithmetic concepts of kindergarten children. In general, the high socioeconomic group was significantly superior to the low socioeconomic group on the instrument used to assess the arithmetic concepts of the group. However, the study revealed a great range in scores of children in the kindergarten attended by the low socioeconomic group.

Dunkley (81) reported a study that compared kindergarten and first-grade students from low and middle class schools on certain number concepts. Concepts investigated included: (a) ability to select a given number of buttons from a heap, (b) ability to mark number symbols, (c) ability to recognize number symbols, (d) rote counting ability, (e) concept of ordinal number. He found that the achievement of pupils in the lower class schools was generally below that of the control classes on the tasks tested. These differences were greater in the first grade than in kindergarten. Again a great range of ability in these tasks was found in the lower class schools. Passy (195) has indicated that the deficit in learning of the lower class student at the third-grade level exists regardless of the particular instructional means used. Unkel's (276) study of achievement in grades one through nine, in which discrepancy scores between actual and expected arithmetic achievement were used, would further indicate that middle and upper socioeconomic class students would, in general, demonstrate greater ability to work up to their measured potential than lower class students.

The research evidence tends to indicate that where no specific intervention in the education of the culturally disadvantaged child takes place, this child will deviate negatively from the middle class achievement norm in arithmetic. Further, this deviation will tend to become greater as the child progresses through school. However, there is much evidence that points to the great variance in achievement demonstrated by "disadvantaged" youngsters. Teachers of these students must be aware of the wide range of ability of these students and refrain from classifying them into one great "deprived" category and teaching as if they were all at one level of development.

How reversible are the cognitive and motivational effects of "cultural deprivation"?

Ausubel (13) in discussing this problem lists some of the effects of a culturally deprived climate. They include: (a) poor perceptual discrimination skills, (b) inability to use adults as sources of information, correction, and reality testing, and as instruments for satisfying curiosity, (c) an impoverished language—symbolic system, (d) a paucity of information, concepts, and relational propositions.

The retarded language development of the lower socioeconomic child has been pointed out by Deutsch (76). According to Ausubel the most important consequence of this retardation in language development is the student's slower and less complete transition from concrete to abstract modes of thought and understanding. (As discussed elsewhere in this book, according to Piaget's stage theory of development, it was noted that this transition normally begins to occur in our culture during the junior high school period.) Prehm (208) has found that verbal pretraining on a conceptual learning task significantly affects the performance efficiency of culturally disadvantaged children. He concluded that both attention to the pertinent aspects of a stimulus situation and verbalization have a significantly positive effect on conceptual performance.

Although much research must be done in this area before making a positive statement, it may be that the depressing effects of cultural deprivation on school achievement in elementary school mathematics may not be irreversible. Ausubel states that effective and appropriate teaching strategies for the culturally deprived child must emphasize these three considerations:

1. The selection of initial learning material geared to the learner's existing state of readiness
2. Mastery and consolidation of all on-going learning tasks before new tasks are introduced so as to provide the necessary foundation for successful sequential learning and to prevent unreadiness for future learning tasks
3. The use of structural learning materials optimally organized to facilitate efficient sequential learning.

Are there differences in achievement in elementary school mathematics between boys and girls?

Sex differences within areas discussed in other sections of this monograph would lead one to answer in the affirmative to the question posed above. Jarvis (144) was interested in gaining evidence on whether boys and girls of similar chronological age and grade placement were capable of doing the same grade level work since their maturational patterns of development seem to differ so markedly. Using about 350 girls and 350 boys at a sixth-grade level, he classified them into three intelligence groups of "bright," "average," and "dull." Results on a standardized achievement test yielded the following findings:

1. The bright boys were found to be superior to their peer group girls in both reasoning and fundamentals.
2. All classifications of male students excelled the female students in their ability to perform arithmetic reasoning functions.
3. All classifications of girls were superior to boys in their ability to execute the arithmetic fundamental operations with the exception of the bright group.

Parsley's (194) findings generally agreed with those stated by Jarvis.

The evidence would suggest to the teacher that boys will achieve higher than girls on tests dealing with mathematical reasoning, while the girls will achieve higher than boys on tests of computational ability. These findings are not as clear at either extreme of the I.Q. range.

Does a history of moving from one school to another adversely affect a pupil's achievement in arithmetic?

Mobility is a characteristic of American society. Does high mobility have a detrimental effect on the achievement of students in elementary school mathematics? Studies that have attempted to gather evidence on this problem generally indicate that mobility does not have an adverse effect upon academic achievement (87, 99, 197, 281). This seemed to be true for either girls or boys and also for students at various intelligence levels. Two of these studies of particular populations indicated that the mobile student achieved at a higher level than the non-mobile student.

Do elementary school students have definite attitudes about elementary school mathematics?

Some studies have been carried out which ask children to indicate their likes or dislikes for school subjects. The reactions of the children are usually construed as indications of positive or negative attitudes toward a particular subject in relation to other subjects taught in the elementary schools. These studies have generally indicated that the students will cluster on either end of the "like"-"dislike" dimension in regard to elementary school mathematics, with relatively few having neutral feelings about the subject.

Sister Josephina (148) asked 900 fifth, sixth, seventh, and eighth graders to select their 3 best liked and 3 least liked school subjects. On ranking the subjects as to number of indications by students as "best liked," arithmetic was ranked in the top three at each grade level. When ranking school subjects as to number of indications by students as "least liked," arithmetic also ranked in the top three. In a similar study of fourth, fifth, and sixth graders in California, Roland and Inskeep (219) found arithmetic to be ranked first in indications by students as the subject liked most; arithmetic ranked fifth (out of ten subjects) in indications by students as the subject disliked most. Arithmetic was last in a ranking of school subjects that had not been indicated by students in their indications of "likes" and "dislikes." Faust (88), in studying more than 2,500 upper elementary school students from Iowa found that pupils prefer the "skill subjects" in the following order: arithmetic, reading, spelling, and language. In a more limited study, Fedon (89) found definite positive attitudes were being expressed by some students and definite negative attitudes being expressed by other students toward elementary school mathematics as early as the third grade.

Findings would indicate that elementary school teachers can be quite confident that some students have definite and relatively strong positive attitudes toward elementary school mathematics, while others will have definite and relatively strong negative attitudes about the subject.

Are attitudes toward elementary school mathematics related to achievement in elementary school mathematics?

Bassham *et al.* (17) used the Dutton Scale to investigate the relationship between pupil attitude toward arithmetic and pupil achievement in arithmetic, with individual differences in mental ability and reading comprehension held constant. In the sample of 159 sixth-grade pupils, more than 4 times as many pupils with a negative attitude toward arithmetic were classified as 0.65 of a grade below expected achievement as were classified as 0.65 of a grade above expected achievement. Almost 3 times as many pupils with positive attitudes overachieved 0.65 of a grade as underachieved that amount. Other studies (88, 168, 235), using varying means of measuring attitude, have generally found a positive relationship between attitudes toward arithmetic and achievement in arithmetic.

What are some factors that seem to influence the development of attitudes toward elementary school mathematics?

Poffenberger and Norton (207) used the questionnaire and personal interview techniques to explore factors in the formation of attitudes toward mathematics. Three hundred thirty-five college freshmen made up the sample used in the study. Two subgroups, one positively oriented toward mathematics, the other negatively oriented, were distinguished. Both subgroups were comparable in ability, support from parents, and general parental expectation. The analysis of the data dealt with sex differences, parental influence, and teacher influence on the development of attitudes toward mathematics. Sex did not seem to be a strong factor in the development of positive attitudes. As many females as males had strong liking for mathematics. They did find, however, nearly twice as many girls as boys who had a strong dislike for mathematics.

Cleveland (64) found no significant difference between boys' and girls' attitudes toward mathematics at the sixth-grade level using the Dutton Attitude Scale.

Parents played an important role in attitude formation, according to the Poffenberger study. There was a difference in parental expectation, as regards mathematics achievement, between parents of the positively oriented and negatively oriented groups. The fathers' attitudes were more influential than the mothers', a finding corroborated by Faust's study at the upper elementary school level. In a further analysis, Poffenberger found that those fathers having a close relationship with their children were most influential. Sixty percent of these "close" fathers who were reported as liking mathematics had children who liked mathematics. Seventy-eight percent who disliked mathematics had children who disliked mathematics.

The teacher's effect on attitudes was difficult to ascertain according to the Poffenberger study, since many of the students could not clearly remember much about teachers in the early grades. Limiting the questions to teachers the students had had for algebra in their secondary school experience, Poffenberger concludes:

> The evidence from the study indicates that self-concepts in regard to mathematics ability are well established in the early school years and that it is very difficult for even the best teacher to change them in spite of the fact that potential ability is much in evidence. Students with an initially negative attitude toward mathematics may go into the classroom with a mental attitude set against the subject which may be maintained even when positive identification with the teacher is made.

From an earlier pilot study (206), the same two investigators had concluded, in regard to early teacher influence:

> Arithmetic and mathematics teachers can have strong positive or negative effects upon students' attitudes and achievement in these areas:
>
> a. They build upon attitudes established by parents.
>
> b. The enthusiastic teacher leads students to like his subject.
>
> c. The teachers who tend to affect students' attitudes and achievement positively have the following characteristics: a good knowledge of the subject matter, strong interest in the subject, the desire to have students understand the material, and good control of the class without being overly strict.

Another factor examined was the relationship between grade level and attitude toward mathematics. Faust found that children's attitudes toward specific school subjects vary from grade level to grade level. Dutton (82), using an attitude scale he constructed, found that grades 5 and 6 were the most crucial in the development of attitudes.

A tangential finding, in a study carried out by Haskell (126) with First Form students in an English secondary school, suggests another factor that may influence attitude. Sociometric grouping of students seemed to positively affect attitudes toward a geometric learning task. The findings suggest that the groups of children in the study, who were arranged in the classrooms according to their preferences for other children, evidenced a more favorable attitude to geometrical drawing than those groups arranged without regard to their wishes.

Is there an association between anxiety and mathematical learning?

It is beyond the scope of this monograph to examine closely the complex issues associated with anxiety and the learning process. The reader interested in a comprehensive examination of "anxiety" may start with Reubush's (223) detailed analysis in the Sixty-second Yearbook of the National Society for the Study of Education.

Studies dealing with mathematics and anxiety fall into two general methodological categories. One set of studies has dealt with the somatic (body) expressions which are manifested by individuals when confronted with stimulus situations of a mathematical type. A second, more commonly used, technique utilizes the paper and pencil questionnaire. Subject's responses to introspective questions are judged to be indicators of anxiety level. These measures of anxiety are then correlated with measures of achievement in mathematics.

Hess (130) presented mental arithmetic problems of varying difficulty to volunteers and then obtained a continuous trace of the size of the pupil of the eye. He found that, as soon as the problem is presented, the size of the pupil begins to increase. It reaches a maximum size as the subject arrives at his solution and then it immediately starts to decrease, returning to its base level as soon as the answer is verbalized. Milliken and Spelka (182) found that freshmen in college with relatively low scores on the mathematics part of the ACE examination tended to exhibit: increased breathing depth and irregularity of depth in their breathing, elevated blood pressure, increased heart rate, and greater and more variable psychogalvanic response deflections when presented with a mathematical task.

Studies which utilized the paper-and-pencil questionnaire as a measure of anxiety (92, 175, 177, 199) have generally shown a significant negative correlation between achievement in arithmetic, as indicated by scores on standard achievement tests, and anxiety as measured by these instruments. This negative relationship suggests that high anxiety is associated with low scores on arithmetic achievement tests. Biggs (25) concluded, after examining research on anxiety and learning in mathematics:

> In arithmetic and mathematics, the inhibition produced by anxiety appears to swamp any motivating effect, particularly where the children concerned are already anxious; or to put it another way, anxiety appears to be more easily aroused in learning mathematics than it is in other subjects.

The teacher can feel quite confident that there is some association between arithmetic achievement and anxiety as measured by the two types of instruments named above. This association, however, appears to be quite complex and any general statement is clouded by factors discussed briefly in the following section.

What are some factors associated with anxiety in mathematics learning?

The trend in studying anxiety and its effects on learning has been a movement away from the study of "anxiety" toward a study of "anxieties." This approach suggests that an individual could be quite anxious about one part of his school experience and less anxious about other parts.

Dreger and Aiken (80) carried out a study with college freshmen to see whether a syndrome of emotional reactions to arithmetic and mathematics could be detected that could be labeled "number anxiety." They concluded from their studies that:

1. Number anxiety does appear to be a separate factor from "general anxiety," although the 0.33 correlation indicates some causal relation probably exists.
2. Number anxiety does not seem related to general intelligence.
3. Persons with high "number anxiety" tend to make lower mathematics grades.

In Milliken's (181) work with college freshmen, he predicted that students who indicated mathematical deficiency would effect greater blood pressure increases under the mathematical stress conditions than those who indicated high proficiency in mathematics with a deficiency in verbal ability. He found that students who had exhibited mathematical deficit did increase in anxiety under stressful mathematics testing for both sexes, as contrasted with a slight increase during the verbal testing. Yet the mathematically able males also reacted with greater physiological change during mathematical testing than during verbal testing. The females were only slightly more anxious in the mathematical testing.

Another specific anxiety that has been studied is Test Anxiety (225). Correlational studies carried out between test anxiety and achievement in elementary school mathematics, as indicated by standardized test results, indicate a rather consistent tendency for children with high levels of test anxiety to perform more poorly than children with low levels of test anxiety.

Sarason (224) reported a stronger negative correlation between level of anxiety and reading test scores than between anxiety and arithmetic test scores for children in grades 2 through 4. Stevenson's (253) results in grades 4 and 6 indicated no tendency for the correlations to be higher on any one particular achievement test. Another finding using the Test Anxiety scale is that girls usually exhibit higher anxiety than boys, although boys may be more defensive about admitting their anxiety. In a cross-cultural study, Sarnoff (226) found that the sex and grade correlates of test anxiety were of about the same order among English school children as among an equivalent group of American children.

A generalization about this complex topic is difficult to make; however, the teacher can be quite confident that high anxiety does have a debilitating effect upon achievement in elementary school mathematics. A selected set of studies (224, 253, 226, 295) would indicate, however, that the degree of the relationship between anxiety and achievement will be affected by such variables in the instructional process as: abstractness of the material to be learned, familiarity with the material to be learned, grade-level of the student, sex of the student, socioeconomic status of the student, as well as the type of cognitive processing (convergent-divergent) required in the task.

Is there a relationship between emotional disturbance in students and arithmetic disability?

Graubard (117) tested 21 children receiving residential psychiatric treatment. Educational disability was measured by comparing mental age with reading and arithmetic ages as ascertained from standardized achievement tests. The mean mental age was 12.82, while the mean arithmetic age was 9.62. None of the subjects were at their expected achievement level in arithmetic computation. There was no significant difference between reading comprehension achievement and arithmetic computation achievement.

Schroeder (231) studied groups of children classified into 5 categories of emotional disturbance in an attempt to find whether there are differences in school skills among groups. The categories were: (a) psychosomatic problems, (b) aggressive behavior, (c) school difficulties, (d) school phobia, (e) neurotic-psychotic personalities. The results indicated that children who are emotionally disturbed are not one group who share the same condition which yields to one program of treatment. Other findings indicated:

1. Variation between behavioral categories was wider in arithmetic than in reading.
2. Mean scores were consistently lower in arithmetic than in reading for all 5 categories.
3. The "school difficulties" category had the lowest mean achievement level in arithmetic and reading.
4. The "neurotic-psychotic" category had the highest mean achievement level in arithmetic and reading.

Evidence would suggest a definite relationship between students with emotional problems and those with arithmetic disabilities. No evidence has been gained, however, on whether arithmetic disabilities are a causal factor in emotional disorders or vice versa. Also, the teacher should be aware that students cannot be collected into one "emotionally disturbed" class and be expected to reflect the same learning disabilities.

Is a student's self-concept related to his achievement in elementary school mathematics?

Bodwin (28), in a study using third- and sixth-grade students, found a positive and significant relationship (.78 on the third-grade level and .68 on the sixth-grade level) between immature self-concept and arithmetic disability. The relationship between immature self-concept and reading disability was somewhat less than the relationship between immature self-concept and arithmetic disability. Disability in both reading and arithmetic indicated a higher relationship with the immature self-concept than disability in other school subjects.

A student's self-concept may be heavily influenced by the opinions others hold of him. Accordingly, Hudgins and Loftus (137) examined the "visible" and "invisible" child in the arithmetic class. Their general hypothesis that "visible" and "invisible" pupils experience different patterns of interaction with the teacher received little support from their data.

The results of a study by Fink (94) appear to confirm the hypothesis that a relationship does exist between adequacy of self-concept and level of academic achievement. This conclusion appears to be unquestionable for boys, considerably less so for girls. It appears that, for whatever reasons, the psychological burden for the male is heavier than for the female.

In a study by Shaw and Alves (236), the existence of a difference in the general perceptual mode of male underachievers and female underachievers was confirmed. The negative perceptual attitudes of male underachievers appeared to revolve primarily around themselves while the negative attitudes of female underachievers appear to be centered on the perceptions of others of themselves.

From a clinical point of view the relationship between adequacy of self-concept (how a child perceives himself) and achievement in elementary school mathematics is a two-way street. For some children the cause of underachievement may be an inadequate concept of self ("I never could do anything well"). For others, a history of failure (real or imagined) in mathematics may be the cause which results in an inadequate concept of self.

The teacher can be quite confident that skillful teaching in a diagnostic sense will bring about for many children improved achievement in elementary school mathematics which will result in an improved, healthier concept of self. And this healthier concept will then permit still greater achievement.

What are some other personality dimensions that may have an effect upon learning in mathematics?

Hebron (127), in examining his data on relationships between learning a new number system and certain personality dimensions, suggested that "extrovert" attitudes in learning may favor the assimilation of the first elementary facts of a new situation, while "introverts" may be more capable, when this stage is passed, in applying these facts in more complex problems. He goes on to suggest that facility in the new quantitative field is most enjoyed by the extrovert, while the introvert is more at ease operating with familiar symbols of a verbal nature.

Levy (167) and Plank (205) have suggested that the over-protected child will not do as well in arithmetic as in other subjects. Rose and Rose (216), using larger samples and homogeneous and heterogeneous social groupings, found no support for the over-protection hypothesis as a whole. However, their data suggest that the variable of over-protection is more likely to become operative in the socially homogeneous classroom than in the heterogeneous classroom arrangement.

Kagan *et al.* (151) have been concerned with conceptual tempo of children. They have found that impulsive children will tend to report the first hypothesis that occurs to them, and this response is often incorrect. The reflective child, on the other hand, delays a relatively long time before reporting a solution and is usually correct. The investigators point out that arithmetic, social studies, and science all require the child to make inferences. Programs that emphasize the discovery method of instruction insist that practice in inference is the key aim of the educational enterprise. Many teachers believe an incorrect inference shows insufficient knowledge and usually do not appreciate the role of an impulsive attitude in determining the quality of the inferential process. Kagan and his colleagues suggest that it may be profitable to consider training the child in reflection when facts and rules are introduced in order to facilitate the general quality of a student's performance.

In studying the dogmatic person and his critical thinking ability, Kemp (155) found the "high dogmatics" have the greater percentage of errors in problems which required the studying of several factors or criteria for decision and the deferring of a conclusion until each factor has been judiciously considered. He writes that apparently the "high dogmatic" has difficulty in tolerating ambiguity and is thus impelled toward a "closure" before full consideration is given to each piece of contributing evidence. This sometimes results in the perceptual distortion of facts and in a conclusion which does not encompass all elements of the problem.

Part Three

Studies Concerning the Learning Environment

How can we best group children for learning mathematics?

The question could also be asked: "Does ability grouping increase learning in mathematics?" Or it could be asked: "Does decreasing the range of ability in an instructional group result in increased learning?"

It is a commonly held belief among school personnel that reducing the heterogeneity, or increasing the homogeneity of a group of children will make it possible for the teacher to bring about a closer fit between the students' ability to learn and the learning experiences. Administrative attempts over the past century have been identified by such expressions as: grade grouping, heterogeneous grouping, homogeneous grouping, ungraded grouping (one-room rural school), X Y Z grouping (by levels of intelligence), "Vestibule" groups, Winnetka Plan, Hosic Cooperative Group Plan (this plan requires teachers to work in small cooperative groups under a group chairman), the Dalton Plan (in which the work was assigned by "contracts"), Platoon grouping, Dual Progress Plan, ungraded primary grouping, ungraded intermediate grouping, departmental grouping, inter-grade ability grouping, and several others.

Far more numerous than the names of the plans are the research studies comparing progress under one plan with progress under some other plan. Shane (234) summarized the findings of most of the studies this way:

> It seems reasonable to conclude that the "best" grouping procedures are likely to differ from one school to another, the most desirable practice often being dependent upon such factors as: (a) the competence and maturity of the local staff, (b) the nature of the physical plant, (c) the school size, (d) class size, (e) the local curriculum or design of instruction, and (f) a highly intangible quality—the intensity of the desire of a teacher or a group of teachers to make a particular plan work effectively.

The philosophy and ability of the able teacher are undoubtedly more important than any grouping plan, however ingenious it may be, with respect to creating a good environment for teaching and learning.

Perhaps the most substantial and significant study of the effects of ability grouping in recent years is that of Goldberg, Passow, and Justman (113). About 2,200 children in 45 elementary schools in the New York City area were studied over the two school years, grades 5 and 6. In addition to academic achievement measures the researchers gathered data from teachers' ratings of students, from students' ratings of students, and students' attitudes toward school.

It is commonly believed that narrowing the ability range of a group of children will make it possible for the teacher to make better differentiation of either method or content. Contrary to this belief, this study reports that simply narrowing the ability range does not necessarily result in better adjustment of method or content and does not necessarily result in increased achievement.

When the data were analyzed for the slow children only, it was found that a single teacher who is capable of working with such children could achieve comparable growth in all areas. But, for the gifted children, no single teacher seemed to be able to provide equally challenging learning in all subjects.

> The general conclusion (of the study) is that, in predominantly middle-class elementary schools, narrowing the ability range in the classroom on the basis of some measure of general academic aptitude will, by itself, in the absence of carefully planned adaptations of content and method, produce little positive change in the academic achievement of pupils at any ability level. However, the study found no support for the contention that narrow-range classes are associated with negative effects on self-concept, aspirations, interests, attitudes toward school, and other non-intellective factors. Therefore, at least in schools similar to those included in this study, various kinds of grouping and regrouping can probably be used effectively when they are designed to implement planned variations in content and method. The administrative development of students must, therefore, be tailored to the specific demands of the curriculum.

In the light of the great amount of research on the effectiveness of various ways of grouping children for instructional purposes, school personnel can feel highly confident that any teacher will teach best in that type of grouping of children in which he has the greatest confidence and sense of security. In a word, until some better plan comes along, teachers will tend to teach best when they are teaching the way they like best.

Does class size affect student achievement in elementary school mathematics?

The few studies that have been carried out that deal with this question would tend to favor smaller classes where quality learning in elementary school mathematics is the desired outcome. In a study carried out in the San Diego school system (210), 36 classes at the first-, third-, and fifth-grade levels of three different size categories were compared for achievement. The evidence suggested that small class size favored achievement in arithmetic at the first- and third-grade levels, but they found no significant differences at the fifth-grade level. Size categories used were:

Small classes:	Grade 1	25-28
	Grade 3	26-29
	Grade 5	29-31
Medium classes:	Grade 1	30-32
	Grade 3	32-34
	Grade 5	34-36
Large classes:	Grade 1	36-39
	Grade 3	38-41
	Grade 5	38-41.

Mennite (180) in studying achievement in parochial elementary schools found some evidence of a significant difference in achievement in mathematics in favor of small classes for the below-average and average pupils. The achievement of the upper IQ groups showed no significant differences between classes of various sizes.

The great amount of variability in the findings of research would indicate that high or low achievement can be obtained at all levels of class size, within reason. Small classes do not automatically bring about significant increases in achievement; however, they do tend to allow the knowledgeable and sensitive professional teacher to operate more effectively and efficiently within the classroom. This is particularly true when the teacher aims for the attainment of a set of noncognitive as well as cognitive outcomes.

Does the ratio of time allotted to the development of meanings and the time allotted to practice during a class period affect learning in mathematics?

Two studies have been reported that are concerned with the problem of determining the optimal time ratio between developmental work and practice work during the typical class period in mathematics.

Shipp and Deer (237), using students at three levels of ability, attempted to determine whether varying the percent of class time spent on developmental activities and on practice work affects achievement as measured by an arithmetic achievement test. This test consisted of: (a) understandings, (b) using arithmetic accurately, (c) solving problems. They concluded:

> 1. There is a trend toward higher achievement, as measured by a general achievement test in arithmetic, when the percent of class time spent on developmental activities is increased.
>
> 2. While the ideal division of class time between developmental activities and practice work could not be determined in this study, it would seem that more than 50 percent of class time should be spent on developmental activities.
>
> 3. The conclusions apply to all ability levels.

Shuster and Pigge (239) used a random sampling of fifth-grade classes to compare three variations of time allotments. One group spent 75 percent of arithmetic class instruction time on developmental-meaningful activities and 25 percent of the time on drill activities; another group split the time 50-50; while another group allotted 75 percent of the time to drill activities and 25 percent of the time to developmental-meaningful activities.

Two parallel forms of a performance test in the addition and subtraction of fractions, which was the topic covered during the experimental period, were administered, one at the end of the experimental period and one as a retention measure 45 days after the particular treatments had been discontinued. Part 1 of this test measured computational skills and to some extent basic fractional understandings, and Part 2 measured verbal problem-solving abilities.

The experimenters found that on the test administered immediately at the end of the treatment period there were no significant differences on test performance. However, the trend was for the groups spending 75 percent of their time on developmental work to rank highest on the test, the 50-50 group to rank second-highest, and the groups spending 25 percent of the time on developmental work to rank lowest. On the retention test there was a significant difference on Part 1 of the test and total test results favoring the 75 percent developmental group. The experimenters summarize their results by saying that children learned skills better by spending less time on drill and more time on developmental-meaningful activities.

The teacher can be quite confident that student achievement is affected by the ratio of class time spent on developmental activities or drill activities. Although the exact ratio is difficult to determine, it appears that at least 50 percent to 75 percent of the time should be spent on developmental activities.

Do children learn more mathematics in good schools than in poor schools?

Another way to phrase this question would be: Will increased educational opportunity improve intellectual achievement? Contrary to commonly accepted "fact," there seems now to be little evidence to support this misconception. That is, it is probably quite true that an increase in educational quality in the form of teachers, books, buildings, and other educational resources will not result in a corresponding increase in educational achievement, desirable attitudes, and aspirations. The Coleman Report (65), the most ambitious study of equality of educational opportunity to date, presents and discusses data collected in a survey of 600,000 children enrolled in grades one, three, six, nine, and twelve of about 4,000 schools that represent a cross-section of all public schools in the United States.

The researchers used tests of verbal ability, reading ability, mathematical and analytical skills, and gathered pertinent sociological information concerning the social composition of the children and their parents, and assembled information on the attitudes and aspirations of the children.

The highest average scores were attained by white children, followed in order by Oriental Americans, American Indians, Mexican Americans, Puerto Ricans, and Negroes.

Variations in the amount of money used to increase quality in the schools have much less effect on the child's achievement than his family background and social environment. That is, a direct increase in the amount of educational opportunity built into the school in whatever form(s) will not result in any appreciable increase in educational attainment.

The student's self-concept is a very significant factor in his academic achievement. The Negro student who has an adequate self-concept, who believes he can control his environment and his future, will score higher on achievement tests than a white student who feels inadequate and unable to control himself, his social and economic milieu and his future.

The authors conclude:

> The data suggest that variations in school quality are not highly related

to variations in achievement of pupils. . . . The school appears unable to exert independent influences to make achievement levels less dependent on the child's background—and this is true within each ethnic group, just as it is between groups.

What about the readability of arithmetic textbooks?

Studies that have been concerned with the vocabulary of elementary school textbooks in arithmetic generally have pointed up the great variability in number of new vocabulary words introduced at each grade level as well as the rate or pace at which the new words are introduced. Hunt (cited in 54) reported on an analysis of six third-grade books whose aggregate vocabulary was composed of 2,993 different words, of which only 350 occurred in all six books. Similarly, Repp (211) reported on an analysis of five third-grade books whose aggregate vocabulary was composed of 3,329 different words, of which only 698 occurred in all five books. She also reported that the average number of new words per page ranged from 3.98 to 6.78 between the five texts analyzed. The range of actual number of different new words, page by page, went from 0 to as high as 69 different new words on one page. Regarding the technical vocabulary of arithmetic, Hunt reported a total of 306 words, of which only 34 were used in all six books she examined.

Smith and Heddens (246) applied a reading grade-level formula to some of the newer experimental programs in elementary school mathematics. The results of the application of the formula to primary grade materials indicated that average reading grade levels tend to be considerably higher than the assigned grade levels of the materials. Application of the formula at the intermediate level also indicated a readability level considerably higher than the assigned grade levels of the materials. In applying the reading-level formula to five different commercial textbook series, Heddons and Smith (128) concluded that the readability level of the five selected commercial texts seemed to be generally above the assigned grade level. This was not to the degree that was true of the experimental programs, however. They also found a great deal of variation of reading-level both between and within the various textbooks at a given grade-level.

Some studies have attempted to assess the commonality of vocabulary introduced in arithmetic texts and reading texts at the same level. Generally the intersecting set of vocabulary words is quite small. Reed (209) analyzed two basic reading series, grades 1-3, and two basic arithmetic series, grades 1-3. Two hundred seventeen different technical vocabulary words were found in the two arithmetic series. Of these 217 different technical terms, only nine were also introduced in either of the two reading texts. Stauffer (248), in analyzing seven different basic reading series at the primary level, and three different arithmetic books at the primary level concluded:

> . . . even if a child had mastered all the different words presented in *all* of the seven reading series (at a given grade level), he would still need to learn to read at least one-half of the words presented in the arithmetic series in arithmetic class. This means that he would need to be prepared to deal with these words semantically (meaning) and phonetically-structurally (speaking) in order to grasp and deal with arithmetic problems or discussions.

The evidence from the various researches cited would suggest that there is great variation in the vocabulary of various textbooks in elementary school mathematics. Where stress is put on meaningfulness in learning as well as individual discovery of some of the material to be learned, it seems imperative that the student be able to read the textbook(s) with a high degree of competence and confidence. With this in mind, the teacher of elementary school children should be quite sure that he must be a teacher of the reading of arithmetic.

Is the mathematical training of elementary school teachers adequate?

In a direct attack at pointing up the inadequacy of preparation of elementary school teachers in mathematics, Glennon (108) found that teachers in service had mastered an average of 55 percent of the understandings basic to the computational processes taught in grades one through six. Subsequent administrations of the Glennon instrument by other investigators (282, 20, 157) over a period of 18 years generally have produced comparable results. Other investigators using other instruments have generally found the same results (124, 190).

Although Glennon found a zero correlation between years of teaching experience and scores on his Test of Understandings, Todd (271), in a more recent investigation using Glennon's test, found a significant negative correlation between number of years of teaching experience and scores on the test.

Callahan (56), using a test based on a more "modern" approach to the content of elementary school mathematics, also found a relatively high negative correlation (—.46) between scores on a mathematical knowledge test and number of years of teaching experience. This would indicate that the higher scores on the test of mathematical understanding were generally achieved by the people with fewer years of teaching.

In the various studies that have been cited in this section, it has been assumed that good teacher understanding of basic mathematical concepts would be necessary to promote satisfactory pupil growth in arithmetic. Bassham (17) investigated this important relationship and found a significant relationship between teacher mathematical understandings and pupil progress in mathematics for the students who were above the mean in intelligence but not for pupils who were below the mean in intelligence.

In general, more work in mathematics for the elementary school teacher would seem desirable. During the academic year, 1962-63, The Committee on the Undergraduate Program in Mathematics (CUPM) conducted a study of requirements and offerings of mathematics in the preservice education programs for teachers in the elementary schools. Results indicated that 22.4 percent of the respondents required no mathematics of prospective elementary school teachers, and 68.9 percent required the equivalent of four or fewer semester hours of mathematics. Of the schools responding, 55.6 percent offer no mathematics courses specifically designed for prospective elementary school teachers.

The CUPM group made the following recommendations in regard to the mathematics courses at the college level for prospective elementary school teachers (268):

1. A course or a two-course sequence devoted to the structure of the real number system and its subsystems
2. A course devoted to the basic concepts of algebra
3. A course in informal geometry.

Follow-up studies, such as that of Fisher (96), would seem to indicate some progress in increasing the mathematics course requirements for preservice elementary school teachers. Figure 1 indicates the increase in course requirements in mathematics by 78 institutions preparing elementary school teachers for the 5-year period, 1960-1965. However, the preservice preparation in mathematics of elementary school teachers in the United States in 1965 was far below the minimum standards set by the CUPM. (See Fisher, Figure 1, p. 196.)

In summary, it would seem that additional knowledge by teachers in the understandings of the mathematics, qua mathematics, of the elementary school would be desirable. Limited evidence would suggest a relationship between teacher understanding and pupil achievement. Also, the relatively high negative correlations between years of teaching

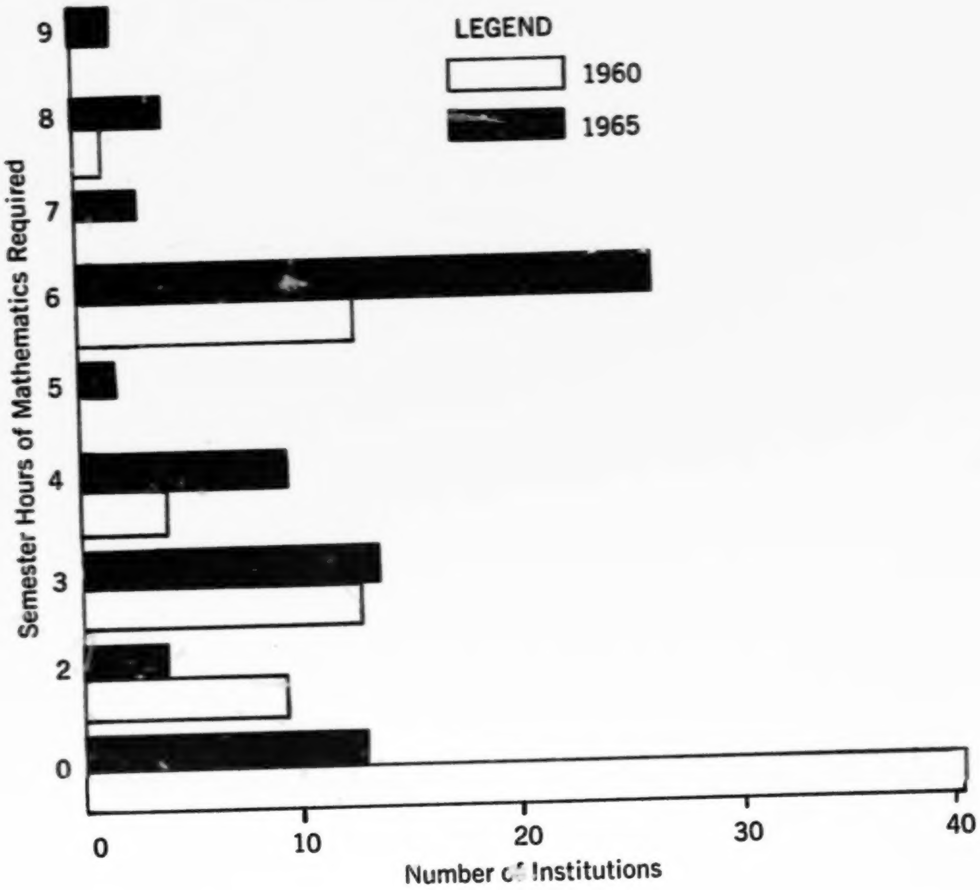

Figure 1. Number of institutions and requirements in mathematics for the preparation of elementary school teachers, 1960 and 1965, from random sample of seventy-eight institutions in the United States.

experience and knowledge of mathematical understandings would suggest the desirability of continuous mid-professional opportunities for teachers to upgrade their competence in the content of elementary school mathematics.

Is the "professional" preparation of teachers of elementary school mathematics adequate?

Whitehead stated in his *Aims of Education* that, "The art and science of education require a genius and a study of their own; and that this genius and this science are more than a bare knowledge of some branch of science or literature" (285). The previous section of the monograph was concerned with the mathematical knowledge of elementary school teachers. This section deals with "professional" knowledge of teachers. What is meant by "professional" knowledge?

Anderson (6) writes:

> It is only as a teacher masters the discipline(s) which bears on his work, as, for example, a physician masters anatomy, that he can be considered to have professional education.

In somewhat the same vein, Melton (179) writes:

> . . . education is to psychology and the social sciences as engineering is to the physical sciences and as medical practice—especially preventive medicine—is to the biological sciences.

Glennon (107) illustrates this interpretation of mathematics education and the disciplines from which it draws in the following way:

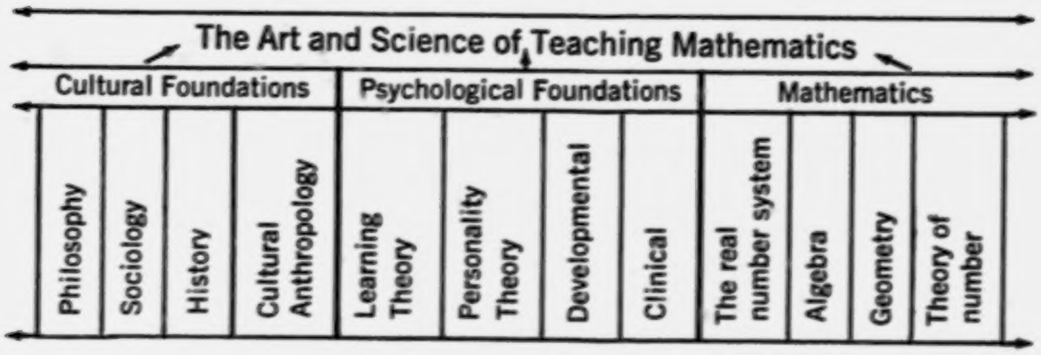
The Art and Science of Teaching Mathematics
Cultural Foundations
Psychological Foundations
Mathematics
Philosophy
Sociology
History
Cultural Anthropology
Learning Theory
Personality Theory
Developmental
Clinical
The real number system
Algebra
Geometry
Theory of number

Figure 2.

He goes on to state:

> . . . the mathematics teacher is not a psychologist as such; nor is he a philosopher as such, a historian, a sociologist, a cultural anthropologist, a clinical psychologist, a personality theorist, etc., as such. But he should have some general competence in several of these basic disciplines. From these disciplines he must draw the principles which help him find answers to his two constant professional questions: *What* mathematics *should* I teach? and *How should* I teach that mathematics to children of varying capacities and personality traits?

Callahan (56) attempted to measure the "professional" knowledge and the "mathematical" knowledge of teachers in-training and in-service. At all three levels—in-service teachers, college seniors completing their work in elementary education, and freshmen who had indicated their desire to become elementary school teachers—the achievement was higher on the "mathematical knowledge" instrument than on the "professional knowledge" instrument.

Thirty years ago, Robinson (215) compared teachers' knowledge of the fundamental principles of arithmetic with their knowledge of methods of teaching arithmetic. He concluded that the professional courses in arithmetic in the professional schools for teachers have been no more successful in eliminating methodological difficulties than they have been in eliminating subject-matter difficulties. There is some evidence that we have made more progress on the subject matter than on the professional knowledge in subsequent years.

Does in-service education have a positive effect on teachers and their students?

Houston and DeVault (136) were interested in three questions regarding in-service work in elementary school mathematics:

1. Does the in-service education program increase the teachers' and their pupils' understanding of mathematical concepts?
2. What was the relationship between the teachers' initial level of understanding prior to the in-service education program and the pupils' increase in achievement?
3. What was the relationship between the teachers' increase in achievement and the pupils' increase in achievement?

They concluded from their study that: (a) The in-service education program was effective in increasing mathematics achievement both for pupils and for teachers; (b) there was a negligible relationship between the teachers' initial mathematical achievement prior to the in-service education program and the pupils' growth in mathematical achievement during the program; (c) growth in understanding of the mathematical concepts of the in-service program was related to pupils' growth in understanding of those mathematical concepts specifically developed in the in-service program.

Ruddell and Brown (222) evaluated three approaches to in-servi work with teachers. One approach was a "one shot" affair in which the consultant spent a full day with teachers before the beginning of classes in September. Another approach spread ten in-service sessions over the year, each session lasting about a half-day. Sessions included a general meeting plus two demonstration classes. Each participant observed about one-half of the demonstration classes. A third procedure made use of "intermediaries." A person from a given school and grade level was chosen to attend the in-service sessions, which included the general session and demonstrations. They then reported back to the teachers in their schools.

Student gains on achievement were measured over a year's time, and it was found that in grades 3, 4, 5, and 6, significant differences between mean gains were shown at every level, and in each instance it favored the "B" group (the group of teachers whose in-service sessions were spread over the year—ten half-days.) The researchers concluded that some type of direct contact between consultant and teacher is necessary to bring about change in teachers' mathematical knowledge and understanding. Furthermore, teachers' knowledge and understanding can be changed just as much from an intense "one shot" program as from a slowly paced, long-range program, but this change is not reflected in the children's achievement.

Another study at the first-grade level gave added evidence that a direct, well-spaced in-service program increases the achievement in mathematical understandings of teachers involved in such a program as well as of the students they teach (221).

The evidence suggests that the resulting increase in achievement of students and teachers from an in-service program depends somewhat on the type of program carried out.

Also, the evidence suggests that teachers who are in the process of changing are more likely to effect similar change or growth in the pupils with whom they work.

Part Four

Studies Concerned with Teaching Method

What are some possible means the teacher can use to motivate students in mathematics?

Sears and Hilgard (233) discuss three types of motives that may be considered: social motives, ego-integrative motives, and cognitive motives.

Social motives have to do with one's relationships with other people. Some teachers may be motivating forces for their students. Amidon and Flanders (4) found that dependent-prone students learned more geometry in the classroom in which the teacher gave fewer directions, less criticism, less lecturing, more praise, and asked more questions than the teacher using a highly direct, lecture method which did not allow for a great deal of participation. Wright and Proctor (294) classified the content of what teachers of mathematics say to their pupils as promoting (a) ability to think, (b) appreciation of mathematics, (c) curiosity and initiative. Peer-related motives may also play an important role. In a study cited within another context (126), there appeared to be increased achievement when students were grouped according to their choice of peers in the group.

Ego-integrative motives can be exemplified by what McClelland (176) terms "achievement motivation." The concept of achievement motivation refers to the need of an individual to perform according to a high standard of excellence. In Lavin's (165) summary of research he found two techniques are commonly used to measure need achievement: projective techniques, and the questionnaire. He concluded from his survey of the research that achievement motivation, as a unitary factor, is not strikingly related to academic performance.

Atkinson (9) combined the factors of anxiety and need achievement in an experiment involving ability grouping. He hypothesized that ability grouping should enhance interest and performance when the achievement motive is strong and anxiety relatively weak. But ability grouping should heighten the tendency to avoid failure when that motive (anxiety) is dominant in the person. The same treatment (ability grouping) should, in other words, have diametrically opposite motivational effects depending upon the personality of the students. Using sixth graders and measures on the Reading and Arithmetic scales of the California Achievement Test, he found that students who were strong in need achievement relative to test anxiety show evidence of greater learning and stronger interest in ability-grouped classes than in control classes irrespective of the level of intelligence. Students low in need achievement relative to test anxiety showed a decrement in interest and satisfaction but no significant change in scholastic performance.

Cognitive motivation refers to motives that reside in the task itself rather than those external to it. Bruner (47) writes, "motives for learning must be kept from going passive in an age of spectatorship, they must be based as much as possible upon the arousal of interest in what there is to be learned, and they must be kept broad and diverse in expression." Within the area of school mathematics, Bernstein (24) writes of two modes of arousal of interest when he states,

> . . . the student who is intrigued by number structure or the commutative law and the student who is intrigued by the use of mathematics in the study of the stock market are experiencing two different kinds of motivational patterns. While it is true that the same individual may experience both of these, it is also possible that the old proverb about one man's meat being another man's poison may often hold true in this type of situation.

Holton (135) investigated the relative effectiveness of four types of instructional motivational vehicles on the achievement of a mathematical task using general mathematics students. The task was couched in four motivational vehicles: (a) automobile, (b) farming, (c) social utility, (d) intellectual curiosity. Kuder preference tests were given to ascertain interests of the subjects. He found significant differences between subjects whose program was related to their indicated interest preference and those whose program was not so related, with the former being more effective in regard to achievement and retention.

Slavina's (242) work in the Soviet Union points up the effectiveness of cognitive motivation and also the restrictions on effectiveness of any type of motivational approach. In his work with 7- and 8-year-olds he found many who exhibited "intellectual passivity." Didactic games involving number calculations were introduced with the object of transforming the subjects' motivation. In his description he writes:

When problems that could not be correctly solved by ordinary means were solved in play, the subject's negative emotions toward mental work began to be replaced by positive emotions and a lively cognitive activity. Initially, however, this new intellectual activity, and the resultant successful solution of the arithmetical problems was confined to the particular play situation and not transferred to school tasks. But by the fifth and sixth day a significant improvement in this direction was noted, indicating that the new cognitive, problem-solving activity, stimulated at first by play, quickly became permanent and was engendered in other than play situations. Nevertheless when an attempt was made to encourage the subjects to use only the more abstract methods of calculation, calling for greater intellectual activity, this was not successful. It was found the subjects lacked the number skills essential to an understanding of addition and subtraction.

The classroom teacher can be quite confident that he and the student's peers, as well as the mathematics itself have some effect upon motivating the student. It may also be the case that students differing in "need achievement" will differ in achievement in mathematics; however, as a unitary construct, there is little empirical support for such a position with the measurement instruments now being used. Used in combination with other construct measures, "need achievement" may be a significant consideration in achievement in mathematics. The teacher must constantly be aware, however, of the necessary readiness skills the child must possess for a particular learning, for, without these, it would appear that even the most highly motivated child will not succeed.

What is the place of "discovery" learning in elementary school mathematics?

A cursory examination of children's texts, professional journals dealing with elementary school mathematics, and other professional publications in the field would seem to indicate a significant increase in popularity of a "discovery" approach to instruction in the past decade. Some adherents of "discovery" techniques in the instructional process discuss the method in a dichotomous sense with a "tell'm and drill'm" method. In reality a pure "discovery" approach and a pure "tell'm-and-drill'm" approach probably fall at extreme ends of a methodological continuum with few adherents at either extreme position.

What would appear desirable is evidence that would suggest conditions under which a "discovery" approach would make the greatest contribution to facilitating the learning process for students, and under what conditions "telling" makes the greatest contribution to the facilitation of the learning process for children. As stated by Cronbach (69), " . . . There is precious little substantiated knowledge about what advantages it (teaching through discovery) offers, and under what conditions these advantages accrue."

Carlow (60), in a concise statement sifted from the literature on discovery learning, lists the following *a priori* claims for discovery learning and the sources for these claims:

1. Discovery methods . . . are the natural and preferred way of learning for man (272).
2. Discovery learning has no parallel for building motivation (129).
3. This rather mystical motivating power is unique to discovery learning (158).
4. Discovery learning promotes better learning and retention (101).
5. Discovery learning leads one to be a constructionist and it avoids the kind of information drift that fails to keep account of the uses to which information might be put (46).

A priori counter claims include:

1. Discovery methods require a vastly increased expenditure of time (12).
2. Beginning with junior high school, understanding is more general, clearer, and more precise when learned through verbal presentation rather than through discovery (12).
3. Failure to discover may result in frustration and superstition (12).
4. Discoveries by children may form closed, idiosyncratic systems because of insights which are circular, repetitious, or merely inane (100).
5. Discovery alone, without adequate consolidation, is likely to be a deceptive and vain pursuit because it is incomplete; and that attempts at discovery prior to age eleven or twelve may lead to "dense fogs of frustrating perplexities . . ." because the learners at that age are unaware of restrictions that logic imposes on possibility and reality (100).
6. Not all children will learn equally well through discovery methods. Thus, the teacher may overestimate the advantages of discovery for the few who are doing the discovering, while ". . . underestimating the cost in loss of communication and bleakness for the many who don't" (100).

Teachers interested in acquiring some knowledge of the complexity of variables associated with learning by discovery would do well to examine the essays on this topic included in Schulman and Keislar (238).

Carlow (60), in a direct attempt at studying the influence of certain variables within the method of individually guided discovery in mathematics, had teachers present a learning task requiring the discovery of 15 generalizations in probability. Subjects were 36 college preparatory ninth graders who had been randomly selected from a larger population. After posing questions, the teachers, working individually with the learners, provided hints in the form of questions until the learner was able to formulate the generalization correctly. Consolidation sessions were held on days immediately following the learning sessions and consisted in practicing only material which had been learned.

A retention-transfer test was administered three weeks after completion of the learning task. Independent variables included: (a) the concepts of ordered partitions and permutations and combinations; (b) the level of consolidation or practice after discovery; and (c) teachers. Organismic variables included: (a) intelligence quotient; (b) conceptual level; (c) certain manifest need factors measured by Stern's Activities Index. Dependent variables included: (a) number of hints required to make the discoveries; (b) time; (c) retention-transfer scores.

Findings suggested that:

1. There was no statistically significant difference between the ordered partitions approach and the permutations-combinations approach.

2. The effects of consolidation were found significant at the 5 percent level. Those students receiving 50 percent consolidation attained mean retention-transfer scores approximately twice as great as those receiving no consolidation.

3. The study also suggested that the personality factors of conceptual level and submissiveness are relatively independent of IQ and very important factors in moderately difficult guided discovery learning.

The limited findings from Carlow's study would suggest to the teacher that consolidation is an important concomitant in the instructional process within a method of guided discovery learning. Also, it was suggested that certain personality factors may be related to successful learning by means of a guided discovery technique. In general, empirical evidence regarding the place of discovery learning in elementary school mathematics is very limited. Kagan warns teachers (150):

> Educators and psychologists must begin to acknowledge the multiple interactions among content, child, and developmental level, for learning in the child is a complex phenomenon. We must begin to develop a patience for elegant answers—and a skeptical irritation toward quick and easy solutions that appear to be a panacea but derive from fragile rational grounds. The task is awesome. But excitement is high and the potential implications unlimited.

What are meaningful approaches to instruction in the primary mathematics program?

There is little "hard" research evidence that would indicate the existence of *one* best approach to meaningful learning at the primary level.

In this section the objective will be to analyze the problem into various methods that have been advocated as effective ways of developing meaningfulness in the elementary school mathematics program at the primary level.

It may be of value to look at the nature of the learnings to be achieved in the primary program. Brownell's insightful analysis helps clarify the nature of some of these desired learnings. He writes (42):

> . . . it is helpful to think of particular facts, concepts, and generalizations as occupying points on a continuum of meaningfulness.
>
> (Zero) . . . ________________________________ . . . N(Maximum)
>
> At the left end of the scale, near the 0-point, are the ideational learning tasks with a minimum of meaningfulness. . . . At the upper end of the scale, near N, are ideational learning tasks which are heavily freighted with meaning. . . . “Two” is an idea which, properly learned, belongs well to the right on the scale of meaningfulness. How much more, then, does “$2 + 2 = 4$” belong near N, involving as it does, not only the idea “two,” but the idea “four,” an understanding of the equivalence (shown by “$=$”) of “$2 + 2$” on the one hand and of “4” on the other.

The numbers 2, 5, $\frac{1}{2}$, etc., are concepts to be meaningfully acquired. Concepts are abstractions. As Clark writes (63):

> To learn the concept of four, or any other number, the learner proceeds from the concrete to the abstract, from things to symbols. Effective learning presupposes that the teacher provide the learner with wisely selected and properly related experiences, and constantly encourage the pupil to generalize, to abstract, to symbolize his responses to them.

What is the most efficient route to travel from things to symbols? What are wisely selected and properly related experiences?

Lovell (169) identifies and discusses three general methods of mathematical concept development. As in many attempts at classification of complex behaviors, there tends to be overlap and seldom does one find in practice a “pure” case of a particular method. For purposes of analysis, however, Lovell's scheme is useful. He cites three general methods: (a) verbal methods, (b) methods based mainly on visual perception and imagery, (c) activity methods.

Verbal methods imply that mathematical concepts build up mainly on spoken and written symbols, in the sense that the child, by manipulating these symbols, comes to comprehend the ideas underlying them. Overzealous application of this approach by proponents of Connectionist Psychology during the early part of the 20th century led to some disillusionment and disfavor with the method. Some contemporary learning psychologists have warned against over-generalizing the ineffectiveness of verbal methods, however. Ausubel writes (12):

> . . . *both* expository *and* problem-solving techniques can be either rote or meaningful depending on the conditions under which learning occurs. In both instances meaningful learning tasks can be related in non-arbitrary, substantive fashion to what the learner already knows, and if the learner adopts a corresponding learning set to do so.

Gagne (101) also points out the efficiency of verbal methods which allow for the "short-circuiting" of more time-consuming inductive techniques, given the necessary antecedent learnings. Since the primary school child may not have a large and varied arsenal of background knowledge with which to cope with verbal methods meaningfully, this method may be less appropriate at this stage than at later stages in the student's cognitive development.

Bereiter and Engelman (22) have recently indicated some success with a direct verbal method in teaching culturally disadvantaged 4- and 5-year-old students. One observer (204) describes a class in the following manner:

> . . . the children started to roar, "eight plus zero equal eight, eight plus one equal nine, eight plus two equal ten, eight plus three equal eleven!"

Evidently these particular children at their level did not find this method the "hateful singsong" that "one and one are two, two and two are four," was to St. Augustine (10) and many others since.

Methods of concept development which are based mainly on visual perception and imagery seek to develop an intuitive cognition by presenting visual perceptual structures. A correspondence is then supposed to arise between the perceptual and physical structures, and the mental structures involved. Some of Stern's (252) writings may aid in illustrating some of this thinking. She writes:

> In the so-called "semi-concrete" approach to numbers, the domino patterns are used in teaching addition.

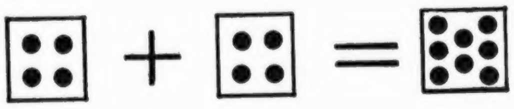

Figure 1.

> The sum is found by counting the single dots. With these patterns, a child can never *see* the equalness of (e.g.) 4 + 4 = 8. In our method, even when we use separate cubes, we show the relation of the parts to the whole. From

his first experiments on, the child constructs the 8-pattern from the subgroups 4 plus 4.

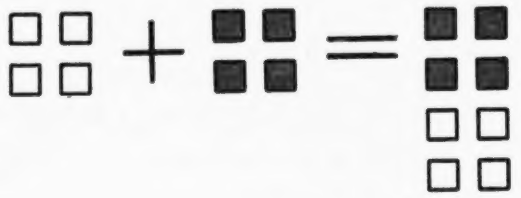

Figure 2.

This shows at a glance how the two addends build up the sum. The structure of the patterns is unforgettable, so that the child can see the subgroups in his mind whenever he reconstructs the picture of 8 and 9, etc.

Suppes (260) suggests three levels of abstraction in the meaningful development of number concepts: (a) a description of sets, [⊘ , ✏]; (b) a consideration of just the number of members of the set, N[⊘ , ✏]; (c) the Arabic numeral representation of number as the final abstraction. Riess (213), however, states that the use of pictures of sets to establish the concept of number in kindergarten and first grade is open to serious doubt. Such use is based on the untested assumption that the child gains his concepts of number through a process of abstraction from groups or collections of objects presented to him.

The action method of number concept formation was popularized by John Dewey (178). Dewey rejects visual perception and imagery as bases of number concepts. Rather, the child's ideas of number are built up by using each number in many different situations that involve him in action. However, Dewey sheds little light on the way in which physical activity is transformed into mental activity.

Galperin (104) in his work at the University of Moscow has developed a theoretical model for the transference of knowledge from physical action to that of purely a mental action. To Galperin, the learning of every mental action passes through five basic states:

1. Creating a preliminary conception of the task
2. Mastering the action, using objects
3. Mastering the action on the plane of audible speech
4. Transferring the action to the mental plane
5. Consolidating the mental action.

The process of teaching a mental action to Galperin then:

> . . . begins with the task of learning something, a task usually set by other people; on the basis of demonstration and explanation, the child builds

up a preliminary concept of the action as seen in the external action of another person. He then makes himself familiar with the action in its external material content, and gets to know it in practice, in its application to things. The first independent form of such activity in the child is, thus, inevitably the external material action.

Next, the action is separated from things and transferred to the plane of audible speech (Slavina [242] describes an approach to this transition using imagery as a necessary intermediate step), where its material foundation is fundamentally changed: from being objective, it becomes linguistic verbal. But the crux of this change is that, from being an action with things, it becomes an action with concepts, i.e., a genuinely theoretical action.

Finally the action is transferred to the mental plane.

Piaget's work, which suggests qualitative changes in concept formation at various stages of cognitive development, has been cited elsewhere. Other approaches that tend to combine perceptual structures with active manipulation in the process of concept development such as that by Dienes (78), Cuisenaire (72), Montessori (184) should be examined by the teacher interested in the process of abstraction.

Much research must be carried out before one can suggest a particular route that is most efficient and effective on the way to the development of an abstraction in elementary school mathematics. It may be the case that there is not *one* most appropriate route for all children. The teacher must be able to recognize the characteristics of pupils' concepts at various ages and stages if he is to understand them adequately and contribute to their growth (270). Hopefully future research in this area will then aid the teacher in his choice of a method that will facilitate the richness of association, accuracy, and precision, which mark the qualitative changes in the emergence of a mathematical concept.

What is the place of practice (drill) in the contemporary mathematics program?

Contemporary programs in elementary school mathematics provide for the attainment of a variety of cognitive skills, abilities, concepts, and understandings as well as for the maintenance of these cognitive learnings. Practice is of the essence in accomplishing the latter objective (maintenance) and is a necessary part of the former (attainment).

Practice has two essential phases according to Burton (51):

> (a). . . the integrative phase in which perception of the meaning is developed; and (b) the repetitive, or refining, or facilitating phase in which precision is developed.
>
> The integrative phase . . . in which meaning is developed demands varied

practice which means many functional contacts and exploratory activities. The refining phase in which precision is developed demands repetitive practice. Varied practice by itself yields efficiency but not meaning. Competent varied practice in early stages will reduce greatly the amount of repetitive practice needed later.

An illustration of these two types of practice might occur in the learning of the addition combinations. During the initial stages of the learning the teacher and children should make extensive use of many and varied manipulative and pictorial materials for the purpose of building the meanings of and relationships among the facts. This would be the integrative phase. Out of this practice would come the systematic arrangement of the addition tables; and further varied practice would result in the development of meanings. Following this careful development would come the repetitive phase of practice with the facts arranged in random order. The purpose of this phase would be the fixing of the facts for efficient recall.

From research studies such as that by Brownell and Chazal (41) has come a major guiding principle in the use of repetitive practice: it must be preceded by a thorough teaching program aimed at the building of *meanings* or *understandings;* or stated otherwise, practice must follow understanding. Weber (284) has indicated that there still is a general misconception by teachers that drill is a way of learning rather than a process for consolidating learning that has been attained during the developmental or integrative stages of learning.

Another section of this monograph deals with the ratio of class time spent on developmental activities compared to practice activities.

Aside from the appropriate positioning of repetitive practice in the instructional process, another consideration focuses on appropriateness of cognitive learnings to which repetitive practice is applied. The basic addition, subtraction, multiplication, and division combinations are examples of learning products of the elementary school mathematics program where a high level of facility with these products is desirable. Therefore, practice both in the attainment and maintenance of these skills is important.

Many contemporary programs in mathematics encourage creative problem-solving activities in an attempt to develop certain process outcomes or objectives. The routinizing of such "process" objectives by drill or practice is quite inappropriate. Luchins' (170) classical experiments point out the rigidity or *Einstellung* effect that is fostered when practice-type activities are applied to creative problem-solving tasks. This result is an antilogy (a contradiction in terms) with the desired outcome of flexible cognitive functioning.

Practice designed to maintain a desired level of functioning for a particular skill is an important consideration in the elementary school mathematics program. Because of the sequential development of a sound mathematics education program, much of the practice on previously learned skills can be "built-in" to subsequently learned materials. This allows the child to use (and therefore practice) skills previously learned, in the development of new learnings. An illustration is pointed out by Capps (59), who found that two groups of students, one group using a common-denominator approach to division of fractions and the other an inversion method, were significantly different at the end of the experimental period in their skill in multiplication of fractions.

One logical explanation that suggests itself would be that since the inversion method of division of fractions requires multiplication as part of the computational procedure, the skills in multiplication of fractions were reinforced. Consequently there was a maintenance of the skill in multiplication of fractions.

The common-denominator method does not involve multiplication of fractions to derive the answer. Thus, there was no opportunity to maintain the skills in multiplication of fractions and computational skill decreased.

The teacher can feel quite confident that practice is a necessary part of the elementary school mathematics program. Wise and discriminating use of practice is important and this involves: its use at the appropriate point, or stage, in the instructional process; its use with appropriate learning objectives of the program; and also differential application to individual children. Some children may only need a small amount of practice to consolidate and maintain high-level functioning, while other children may need a greater amount of practice.

What do we know about diagnosis in arithmetic?

Relative to the whirl of activity taking place in curriculum and certain areas of the instructional program in elementary school mathematics, the area of diagnosis has been quite static. Early work in diagnosis in arithmetic was largely limited to determining the kinds and frequency of errors in computational skills. Later work in this area expanded concern to growth in meanings and understandings basic to the computational processes, growth in problem-solving ability, growth in mental arithmetic, and growth in ability to make quantitative judgments.

Of concern in some of the newer thrusts in diagnosis is the complex relationship between growth in arithmetic development and affective factors such as anxiety, motivation, and attitude. Concerning this point, Bernstein (23), reporting on a survey of articles dealing with remedial procedures in elementary school mathematics stated:

> While some observers have said little, if anything, about emotional and physical factors, the data describing the results indicate that the factors were there and, quite often, efficiently dealt with.

Another indication of the importance of affective factors was indicated by Ross (218). Reporting on the 20 case studies carried out with sixth and seventh graders who had indicated a great deal of disparity between actual achievement and expected achievement in elementary school mathematics, he indicated, among other findings, that 63 percent of the causes of underachievement identified by classroom teachers were of an emotional nature, involving lack of interest, home or school maladjustment, short attention span, or limited initiative. In this study, arithmetic underachievement appeared as a complex and multiple-factored disability.

Wilson (288) comments on the complexity of underachievement in elementary school mathematics when he writes:

> It has become increasingly apparent in our work with individual children . . . that underachievement in mathematics . . . is far from being of one kind. . . . Of several children with the same degree of general underachievement in mathematics, each has unique symptomatic patterns of that underachievement.

With the realization of the complexity of the nature of underachievement, methods of diagnosis must also undergo change. Brueckner (45) suggested four general methods that could be used to analyze errors and faulty methods of work: (a) observation of the pupil at work, (b) analysis of written work, (c) analysis of oral responses, and (d) interviews. The techniques of greatest use in a sound diagnostic program will be those that lean away from the more mechanical types and lean toward the more clinical procedures.

Only through these procedures, such as the interview, can the teacher discover the thought processes, the maturity or immaturity of thinking, that precedes and determines the written work. Where the problem lies beyond being a cognitive one in arithmetic, teachers will need to refer the case to others such as psychiatrists, psychologists, social case workers, etc., whose professional training makes them the more appropriate people to work with the child.

Gagne's (102) structure, using an analysis of antecedent learnings essential in acquiring a particular segment of knowledge in mathematics, may offer the teacher a useful model for diagnosing cognitive problems in elementary school mathematics.

The statement by Brueckner (45) which follows, gives the teacher insight into the scope of ability required in diagnosing arithmetic learning. He says:

> To diagnose arithmetic ability competently, the examiner must have a clear conception of the functions and objectives of arithmetic instruction, must be thoroughly acquainted with the scientific studies of the factors that contribute to success in arithmetic, must know the symptoms and causes of various unsatisfactory conditions, must be able to use effective techniques for bringing to the surface facts concerning the nature of the pupil's disability and his thought processes that would ordinarily be unanalyzed, and must be able to interpret the facts revealed by his study of the pupil and to suggest steps to correct the condition.

How can we measure growth in arithmetic?

The teacher of arithmetic is a teacher of the whole child; he is not a teacher of one "slice" of the child—the arithmetic "slice." It is his responsibility to develop those social and mathematical understandings, attitudes, habits, appreciations, skills and abilities, personal, social, and moral values that are essential for effective living in a democratic society. The teacher must be aware of the total growth of the learner if he expects to guide this total growth. Accepting the responsibility for developing the whole child, the teacher must also accept the responsibility for measuring the whole child. The use of teacher-made and standard pencil-and-paper tests only, which gather evidences of computational skill and verbal problem-solving ability, is an incomplete testing program. Such a program measures growth in limited aspects of the total development of the learner, and does so with limited success.

The modern program of evaluation in arithmetic makes use of a variety of techniques and devices. Also, it is concerned with studying the child's process of learning as well as the outcomes of learning. One of the most fruitful techniques for studying the child's thinking is the interview—used widely in clinical work. Buswell (53) suggested six methods of studying pupils' thinking. Brownell and several of his students have used the interview with brilliant success, contributing much to our knowledge of how children learn arithmetic. Teachers should use the interview much more than they do at present. The need for improved group pencil-and-paper tests to measure understandings has been stated by Brownell (40), Glennon (110), Spitzer (247), Sueltz *et al.* (256), and others.

A complete program of evaluation in arithmetic will measure growth in ability to make judgments in quantitative situations, ability to do mental arithmetic, attitudes toward arithmetic, appreciation of the uses of arithmetic, and other outcomes. It will make use of the techniques and devices mentioned above, and, in addition, will make use of real and contrived problem situation tests, dramatizations, anecdotal records, growth charts, and others. Extended discussions of evaluation of the total elementary school mathematics program may be found in sources 83, 111, and 266.

Does homework help?

This question is much too general to answer with the limited research evidence available. If the objective of the homework is immediate increase in computational skill, there is some evidence that this objective can be achieved by regularly assigned homework in the middle and upper elementary grades (114).

On the other hand, there is little evidence that higher forms of cognitive functioning, such as problem-solving ability, will be affected by regularly assigned homework problems (163).

The kind of homework mentioned in the previous paragraphs is generally given to reinforce or consolidate previous learnings. Another kind of homework may be assigned to enrich learnings that are developed during the school day. Hudson (138) found that sixth-grade students and teachers feel that an individual project is a type of assignment which promotes learning. How these different kinds of homework affect attitudes and interest in mathematics is unclear.

It would seem from the standpoint of research and common sense that the teacher should be aware of the specific objectives sought by a homework policy as well as the difference in individuals to which such homework is assigned. It is reasonable to say that indifferent, routinized homework assignments, imposed by the teacher and opposed by the pupil, bring about little or no growth in desirable mathematical learning.

Should children be allowed to count when finding answers to number facts?

No two children in any grade are at the same level of development in their control over all aspects of number work. Where one child may be able to give a mature, automatic response to a number fact, another child is able to give a response to the same fact only on any one of several less mature levels. When two children seemingly give equally mature responses, further probing may give evidence of a more complete understanding by one child than by the other. Also, any child may give a mature response to one number fact and an immature response to another number fact.

Brownell (37) identified four levels of development from immature to mature in responding to number facts: (a) counting, (b) partial counting, (c) grouping, and (d) meaningful habituation. Whether a child should be allowed to find answers by counting depends on his level of development. In the early stages of learning the facts, he should be allowed, even directed, to find answers by counting and grouping. As he matures, he should approach and attain the level of meaningful habituation.

Beckwith and Restle (21), in their experimentation dealing with the process of enumeration, suggest that there may be differences between children's and college students' use of spatial arrangement in counting. Fairly young children, 7 to 10 years of age, seem to show sensitivity to the organization of the visual field. That is, even when a child is enumerating one by one, he may work rapidly within one group, then pause and consolidate his result in some way, and then attack the next group. The pausing, and the ability to divide the task into suitable parts, is a generally important part of a long serial task. College students seem to make special use of the rectangular array, presumably by using multiplication. For both young children and college students, the rectangular array may facilitate the process of enumeration to a greater degree than a linear, circular, or scrambled presentation of the objects.

We should not expect a child to begin with a mature level of response. Brownell and Chazal (41) concluded that children do not come rapidly to mature thought processes and hence to true mastery of the facts. They move through levels of development from immature to mature.

Dawson (74) found that the use of complicated pictures tended to foster counting and to inhibit development of more mature ways of grouping numbers. The better alternative is to use pictures with no social significance such as dots.

The teacher can feel confident that counting is acceptable behavior for the child in the early stages of learning; he must also accept the fact that his guidance includes helping the child grow from less mature to more mature behavior.

What meaning(s) and what algorism(s) for the operation of subtraction?

Three meanings for the operation of subtraction are generally developed; the "take-away" idea, the "additive" idea, and the "comparison" idea. Gibb (105) reported the thought processes used by second-grade children when solving problems involving subtraction situations, additive situations, and comparative situations. Crumley's (70) study indicated that children tended to see the subtraction process as a "take-away" process regardless of the teaching method used. Schell and Burns (228) found:

1. Children's arithmetic textbooks that they examined for both grades 2 and 3 indicated considerably greater opportunity for work with "take-away" subtraction situations than for other types.

2. Pupils in the study performed best of all on "take-away" subtraction situations and least well on "comparison" situations.

3. The pupils themselves felt that the "take-away" situations were the easiest to work.

4. The pupils' drawings of their thinking of the solutions showed evidence of lack of understanding that the 3 situations, from the standpoint of visual manipulation, are different.

It would seem that thorough teaching of subtraction requires a systematic effort on the part of the teacher to build concepts for the three situational uses of the subtraction concept.

The subtraction algorism has been an object of investigation for many years. Two algorisms, equal additions and decomposition, have received the lion's share of attention. In the equal-additions method (A) 10 ones are added to the 3 ones making 13 ones; 7 ones can be taken from the 13 ones leaving 6 ones. To compensate for the 10 ones added to the 3 ones in the minuend, 1 ten is added to the 2 tens making 3 tens in the subtrahend. Then, 3 tens from 4 tens is 1 ten. In the decomposition method (B) 1 ten of the 4 tens is changed to 10 ones and added to the 3 ones; 7 ones can be taken from the 13 ones leaving 6 ones; 2 tens from the remaining 3 tens is 1 ten.

$$\text{(A)}\quad \begin{array}{r} 4\;{}^{1}3 \\ \underline{-{}^{3}\cancel{2}\;7} \end{array} \qquad\qquad \text{(B)}\quad \begin{array}{r} {}^{3}4\;{}^{1}3 \\ \underline{-\,2\;7} \end{array}$$

Early research studies (see summaries by Ruch and Mead [220], Johnson [147], and Brownell and Moser [43]) show that neither of the two methods was markedly more efficient than the other, but when both were taught in a mechanical fashion pupils who use the equal-additions method had a slight advantage in rate and accuracy.

In a study termed by Cronbach (68) "one of the best-executed of all educational experiments," Brownell and Moser (43) compared the effectiveness of the decomposition and equal-additions methods when each was taught 2 ways—meaningfully and mechanically. The success of the methods was judged not only on the basis of rate of work and accuracy of work, but also on the basis of smoothness of performance, degree of transfer of training, and the values inherent in the use of a crutch in the early stages of learning. Using a variety of data, the researchers found that: (a) the decomposition method when taught meaningfully was the most successful method; (b) the equal-additions method was difficult to rationalize; (c) the use of the crutch facilitated the teaching and learning of the decomposition method; (d) children discarded the crutch when encouraged to do so by the teachers.

The decomposition algorism has had widespread acceptance in this country during the past 15-20 years. However, the development used in some of the contemporary experimental programs may cause some questioning of the advantage found in the decomposition algorism. Cronbach (68) pointed out the development of a technique using the number line which may make the rationalization of the equal-additions algorism more meaningful. The technique has the added benefit of being useful with negative numbers as well as the whole numbers. Whether this technique of rationalizing the equal-additions method would work better or worse than the meaningful decomposition method, we do not now know.

In summary, it would seem desirable for the teacher to be aware of the general superiority of meaningful approaches to teaching the subtraction algorism. The meaningful decomposition method would appear to be the most widely accepted in contemporary programs. The teacher should be aware, however, of the alternative algorisms and be aware of their contribution to "readiness" for future learnings in mathematics.

What method(s) should be used for introductory work in multiplication?

Initial work with the operation of multiplication places much emphasis on developing meaning for the operation as well as familiarity with the "structure" of the operation. Many contemporary programs in elementary school mathematics attempt to achieve meaning for the operation by interpreting it in terms of repeated addition, arrays, mappings, and Cartesian products.

Gray (118) investigated the effectiveness of a program of instruction in introductory multiplication which was based on the development of an understanding of the distributive property. Working with 22 classes of third graders who had no previous formal instruction in multiplication, two experimental groups were randomly set up. Treatment-1 development explained multiplication in terms of repeated additions and arrays of objects in rows and columns. The lessons provided for practice or drill in memorization of the combinations, but made no mention of the distributive property or its applications.

Treatment-2 lessons were identical with the first five Treatment-1 lessons to insure that both groups had the same basic understandings of multiplication through the combinations with two as a factor. The remaining lessons of the Treatment-2 group were designed to introduce and explain all additional multiplication combinations solely in terms of the distributive property for multiplication. There were 18 experimental lessons for each group of approximately 40 minutes each. Pre- and post-tests on multiplication achievement were administered as well as a post-test of transfer, a retention test of multiplication achievement, and a retention test of transfer. Individual interview tests were also administered to a random sample of subjects to ascertain their understanding of the distributive property and the multiplication operation.

Gray concluded from the study:

> 1. A program of arithmetic instruction which introduces multiplication by a method stressing understanding of the distributive property produced results superior to methods emphasizing repeated addition and the array.
>
> 2. Knowledge of the distributive property appears to enable children to proceed independently in the solution of untaught multiplication combinations.
>
> 3. Children appear not to develop an understanding of the distributive property unless it is specifically taught.
>
> 4. Insofar as the distributive property is an element of the structure of mathematics, the findings tend to support the assumption that teaching for an understanding of structure can produce superior results in terms of pupil growth.

The interview test used with the Treatment-2 group indicated that subjects in this group who used the distributive property to find products were significantly superior in IQ to those who relied on rote memory, counting, or repeated additions to find products. Schell (227) found in his work on introductory multiplication that subjects who had been classified as low-achieving on a pre-test of general arithmetic achievement, had considerably more difficulty with the distributive property items on a test than they did with items on other aspects of multiplication. However, the high-achieving pupils performed at an approximately constant rate on both sets of items.

Although much research is needed on the use of mappings, Cartesian product, and more evidence is needed in the use of arrays, and repeated additions, the limited research cited would give some evidence that, for children of average to above-average intelligence, the use of the distributive property provides some benefits in the acquisition of the multiplication combinations as well as transfer to subsequent untaught combinations. This does not mean that repeated additions or arrays are ignored in developing meaning, since it was pointed out that the group using the distributive property in Gray's study were taught the first five lessons using repeated additions and arrays as a rationale for the combinations through two as a factor.

What method of division should be used with whole numbers?

Two kinds of division situations are generally identified, measurement and partitive. Given a set of elements that is to be separated into equivalent subsets, measurement problems are those requiring that the number of subsets be found and partitive problems are those requiring that the numbers of elements in each subset be found.

Gunderson's (121) study suggested that problems based on partitive-type division situations were more difficult for second-grade children than problems based on measurement division situations. Hill's (132) study with upper grade children suggested that these children prefer measurement problems but their performance on the two types was not significantly different.

Zweng (296) introduced a further analysis of division situations by discriminating between "basic" measurement situations and "rate" measurement situations, as well as "basic" partitive situations and "rate" partitive situations. The distinguishing characteristic of a "rate" situation is that two sets of objects are given in the problem. (The interested reader may turn to the next question, which discusses the idea of "rate-pairs.") Examples of the four situations follow:

1. "Basic" measurement: If I have 8 balloons and separate them into bunches of 2 balloons, how many bunches will I obtain?
2. "Rate" measurement: If I have 8 balloons and put the balloons into sacks, placing 2 balloons in each sack, how many sacks will be used?
3. "Basic" partitive: If I have 8 balloons and separate them into 4 bunches, with the same number of balloons in each bunch, how many balloons will there be in a bunch?
4. "Rate" partitive: If I have 8 balloons and put them into 4 sacks with the same number of balloons in each sack, how many balloons will there be in each sack?

Another aspect of this study dealt with the effect of different methods of presenting the problems to the children. All "basic" problems were illustrated with just one set of objects, the set of objects given in the problem. For "rate" problems, which describe two sets of objects, some were illustrated with both sets of objects and some were illustrated with only one.

Some of the findings would suggest that:

1. Partitive division problems are more difficult for second-grade pupils than measurement problems.

2. Partitive "basic" problems are considerably more difficult for second-grade children than partitive rate problems.

3. Overall, division problems presented with one set of objects are more difficult for second-grade children than problems presented with two groups of objects.

4. Most of the difficulty that the children had with problems using one set of objects could be accounted for by the partitive situations where only one group of objects were used. The differences between partitive situations, using two groups of objects, and measurement problems were in no instance significant.

Another interesting observational outcome of this study concerned the manner in which partitive problem situations were solved by children. Two methods of solving these partitive situations were identified: (a) sharing, where the child assigned the same number of elements to each of the required subsets but did not use all elements on the first assignment; (b) grouping, where the child assigned all the elements on the first processing. The children in the study solved the majority of the problems by means of grouping procedures. Children who used a sharing technique seldom used one-by-one sharing, but would choose as their first assignment to each group a number of elements that was over 50 percent of the number of elements required in the group.

Algorisms used in processing division situations generally fall into two main categories; one can be referred to as a "subtractive" algorism, the other as a "standard" algorism. These two algorisms are illustrated on page 91 in their most mature form. Each can be carried out in many less mature ways during developmental stages of learning.

Van Engen and Gibb (279) compared the two algorisms. They concluded:

> 1. Children taught the conventional method of division will no doubt attain greater achievement in solving kinds of problems taught than will the children taught the subtractive method.
>
> 2. The subtractive method of division can be expected to be more effec-

$$\begin{array}{r|l} 16\;\overline{)\;9679} & \\ \underline{6400} & 400 \text{ (Subtracting 400 sixteens)} \\ 3279 & \\ \underline{3200} & 200 \text{ (Subtracting 200 sixteens)} \\ 79 & \\ \underline{64} & 4 \text{ (Subtracting \ 4 sixteens)} \\ 15 & \overline{604} \text{ (Adding the partial quotients)} \end{array}$$

Subtractive Algorism

$$\begin{array}{r|l} 604 & \\ 16\;\overline{)\;9679} & \\ \underline{96} & (6 \text{ hundreds} \times 16) \\ 79 & \\ \underline{64} & (4 \text{ hundreds} \times 16) \\ 15 & \end{array}$$

Standard Algorism

tive in enabling children to transfer familiar experiences to unfamiliar situations where the general context still remains the same.

3. Children taught the subtractive method can be expected to have a better understanding of the idea of division.

4. There is no reason to expect real differences between the contributions of the mental functions of the two methods to retention of skill and understanding that may be achieved. This seems to be more closely tied up with teaching procedure regardless of method of division.

5. Partitive and measurement situations are different for the two groups insofar as the ability to comprehend the situation is concerned. At the end of the fourth year program, one would expect to find that the idea of quotitive (measurement) division is easier for those taught the subtractive method and that the idea of partitive division is easier for the conventional method group.

6. Children of low intellectual ability can be expected to have less difficulty understanding the process of division if they use the subtractive method rather than the conventional method.

7. Apparently method differences in the processing of division problems make little difference in the high intellectual groups, that is, other than might be expected between the two methods groups when variables of intelligence and arithmetic achievement were controlled.

In a small exploratory study, Scott (232) attempted to compare groups of third-grade children who were taught both the standard and subtractive algorisms with groups who were taught only one of the algorismic forms. Teachers of the groups learning both algorisms used the subtractive algorism for all measurement division situations and the standard algorism for all situations of a partitive nature. Inasmuch as the results of the study generally favored the groups of third-graders who had learned both algorisms, the writer suggested that it would be difficult to eliminate the possibility that instruction in division can be aided by the inherent logic of the algorismic form.

The teacher can be quite sure that young children initially introduced to division situations will generally find measurement-type problem situations more understandable than partitive situations. There was some evidence that would suggest that partitive situations using rate-type problems with concrete objects to illustrate both parts of the rate situation made for increased understanding of the partitive-type problem. Contemporary texts generally use either one of two approaches to processing division situations; the subtractive algorism, or the standard algorism. Teachers should be familiar with both algorisms, and be aware of the strengths and weaknesses that can be anticipated from each approach.

What are some factors which may contribute to meaningfulness in work with the fraction program?

What follows is not a series of citations of research reports, but a series of personal opinions on matters which may, or may not, be pertinent to the problem of developing meaning for formal work with fractions in the intermediate grades. The elementary school teacher should be aware of these various positions and, where particularly interested, can go to the sources cited for more detailed reports.

One topic that has gained some interest is the distinction between rate pairs and fractions. Van Engen (278) points out that the child

numbers without involving a mathematical operation. For example:

(a) Bill bought 3 neckties for $2.

(b) Jane and Sue packed apples for Christmas baskets. They placed six apples in each basket.

These situations describe what is called a rate. A rate is a physical situation in which we make a many-to-many correspondence. Van Engen points out that, in general, number pairs representing rates are not added as we usually think of adding fractions. For example:

> I buy ribbon at the rate of five yards for $2. Later I need more of the same ribbon. I find that the price is now 3 yards for $1. What price did I pay for the eight yards of ribbon? Obviously 8 yards for $3.

Because of this characteristic, the rate pair is not considered a number. However, the fraction is considered a number. The various mathematical properties of rate pairs and fractions are then compared. Implications for work with rate pairs in curriculum decision-making are then presented.

Crumley (71) discusses the abstraction process from rate situations to the concept of ratio. He points out that any expression of a rate involves 2 numbers and 2 units of measure. When we pull just the number pairs from a rate situation, we ignore the units of measure. The result is an abstraction which can be recognized as a ratio. It is interesting and instructive to notice that the abstraction of rate situations to get a ratio is similar to the abstraction of groups of objects to get a number.

Mueller (185) discussed in some detail the number-numeral distinction as it relates to fractions. Implications from the position of "a fraction as a numeral" are discussed, with specific emphasis being placed on this development and its relation to problem solving in elementary school mathematics.

Another topic receiving attention is that of the appropriate order in presenting "common" fractions and "decimal" fractions in the learning sequence. Johnson (145) had cited some reasons for teaching "decimal" fractions before "common" fractions. More recently Riess (213) has suggested:

> Since the notation used for decimal fractions is the logical outcome of the notation for whole numbers, students in the intermediate grades should begin their formal study of fractions with decimal fractions rather than with a review of common fractions. When fractions are introduced to them as an extension of whole numbers, their capacity for independent and creative thought can be activated. They can figure out for themselves how to compute with decimal fractions deducing methods of computation with these fractions from an analogy with whole numbers.

What method should be used when dividing by a fraction?

The problem of dividing by fractions has been an object of much discussion and some investigation. One object of research has involved the comparison of the "common-denominator" method of dividing by a fraction with the "inversion" method. Brooke (33) reported a study of the common-denominator method for introducing division of fractions. He found the common-denominator method to be more successful than the inversion method if the examples used involved no remainders in the answer. Stephens (250) found no significant difference between the skill attained by the students who learned either the "common-denominator" or "inversion" methods. In a follow-up study, she also found no difference in retention of the skill by either group (251).

One variable that has received some attention is that of the "meaningfulness" of the inversion method. Sluser (243) attempted to test the hypothesis that teaching the division of fractions with an explanation of the reciprocal principle as the rationale behind inversion would be more effective than teaching the division of fractions through the manipulative device which simply involves inverting the divisor and multiplying. Using 300 sixth-grade subjects stratified into 3 IQ levels, he concluded: (a) instruction in division of fractions with an explanation of the reciprocal principle as the rationale behind the inversion method was less effective than instruction by the inversion method which simply taught the pupils to invert the divisor and multiply; (b) students whose IQ was 121 and above could comprehend the mathematical principle behind inversion and this instruction tended to improve their understanding of and skill in the operation of division of fractions; (c) there was evidence that the average and below average students could not comprehend the mathematical principle involved and this instruction resulted in more confusion concerning the operation of division of fractions.

The work of Capps (59), which was mentioned in another section, points up the differential effects of the "common-denominator" and "inversion" method of dividing by fractions on skill in multplying by fractions.

The findings of research are far from being clear in this area. The increased concern for "meaning" seems to have given some upsurge in the use of the "common-denominator" method. However, various techniques using both perceptual and mathematical structures have been advanced to aid in developing meaning for the "inversion" method. There is some limited evidence to suggest that average to below-average intelligence students do not grasp this meaning as rapidly as the more intellectually capable students and may become confused if consolidation is attempted prematurely. Also, teachers who plan to teach the "common-denominator" method should make special provision for maintenance of skills in multiplication of fractions.

How can we improve ability to solve verbal problems?

One of the important objectives of the elementary school mathematics program is the development of the ability to solve verbal problems. It is through the provision of large and well-ordered amounts of experience with verbal problems within a sound textbook program that the child develops ability to solve arithmetic problems and transfers this ability to solving similar problems occurring in out-of-school, real-life situations.

Studies of verbal problem-solving ability such as those by Kliebhan (161), Emm (85), Alexander (2), Martin (174), Engelhard (86), generally attempted to isolate certain factors that contribute to success in verbal problem solving in arithmetic. Buswell (55) used tests and recordings to get at the thought processes of a group of high school and university students as they attempted to solve six sets of problems. From the evidence gathered in these studies, it would appear that the following factors contribute to success in verbal problem solving.

1. General reading skill, including a knowledge of word meanings and of words used singly, in phrases, and in sentences.
2. Problem-solving reading skills, including:
   a. Comprehension of statements in problems
   b. Selection of relevant details in problems
   c. Selection of procedure to solve problems.
3. An arithmetic factor, which includes computational skills in which the pupil understands *when* to use a process as well as *how* to use it, and also a mathematical understanding whereby the pupil has meaningful concepts of quantity, of the number system, and of important arithmetic relationships.
4. A spatial factor, which involves an ability to visualize and think about objects and symbols in more than one dimension and the use of mental imagery to help clarify word meanings when making comparisons and judgments.

Additionally, Buswell found considerable variability in sequence of operations. He also found that problems expressed in letter symbols gave subjects more difficulty than similar problems expressed in numbers.

In general, a rather high relationship would exist between the various factors that have been mentioned. No doubt a very complex interrelationship exists between these factors (and others not mentioned) and ability to solve arithmetic problems in a verbal context.

Some investigators of this complex problem have carried out experimental studies in order to try to ascertain the effects of some specific training procedure on ability to solve verbal arithmetic problems.

Various studies (123, 146, 274) have suggested that the study of mathematical vocabulary should be an important part of instruction in the area of verbal problem solving in arithmetic. Vanderlinde (277) compared the achievement of experimental groups of fifth-grade students who were given direct study experiences of some 200 technical words, word groups, and symbols with a similar group of fifth graders who were given no such direct study experiences. Results indicated that individuals in classes in which direct-study techniques were used achieved significantly higher on a test of arithmetic problem solving than did individuals in classes in which no special attention was devoted to the study of quantitative vocabulary. The direct-study of quantitative vocabulary was significantly more effective with pupils who have above-average and average intelligence than with pupils who have below-average intelligence. The experimental group also achieved significantly higher scores on a test of arithmetic concepts than did their counterparts in the control groups.

Pace (192) attempted to determine the effect of understanding of the four operations—addition, subtraction, multiplication, and division—upon problem-solving ability of fourth-grade students. During periods of systematic instruction, children in the experimental group were asked to read the problems, tell how they were to be solved, and then defend their choice of process. Emphasis was upon *how* the problem was to be solved and *why* a given process was appropriate. The control group in the study merely solved the sets of problems, identical to Group I, but there was no discussion of the work. Standardized instruments as well as interviews were used in evaluating results.

It was found that both groups showed improvement on "conventional" type problems; however, the experimental group showed greater improvement than did the control group. The interview evaluation indicated that both groups showed an increase in number of correct solutions to "conventional" problems based upon mature and immature understanding; however, the experimental group showed a greater increase than did the controls. With problems on the measurement instruments which contained "distorted cues," both groups showed improvement in number of correct processes and procedures; however, the experimental group showed greater improvement than did the controls.

Results of the study would suggest that children show gains in problem-solving ability if they are merely presented with many problems to solve, but they show even greater gains if systematic instruction for the purpose of developing understanding of the four processes is provided by the teacher. Irish's study (141) would also suggest that where students are given opportunities to develop systematically their ability to generalize the meanings of the number operations and the relationships among these operations and develop ability in formulating original statements to express these generalizations, the result will be increased ability in solving verbal problems in arithmetic.

Wilson (289), using fourth-grade subjects and one-step addition and subtraction problem situations as a vehicle, compared two specific problem-solving approaches. Program A attempted to create a mental "set" in the subjects which called for a focusing on the sequence of the actions and events in the verbal problem situation. Essentially, Program A involved training the subjects to:

1. "See" or recognize the real or imagined action-sequence structure of a problem.

2. Express the action-sequence in an equation.
3. Compute using the operations indicated by a direct equation.

Program B attempted to create a mental "set" in the subjects which called for a focusing on the "wanted-given" relationship in a problem. Essentially, Program B involved training the subjects to:

1. Recognize the wanted-given relationship embedded in a problem.
2. Express the wanted-given relationship in an equation.
3. Compute using the operation directly indicated by the equation.

Under Program A when a child is faced with a verbal problem he presumably "sees" the action-sequence structure of that problem. His choice of operation would be based on his recognition of the commonality of that structure's attributes with those action-sequence attributes of one of the operations. Under Program B, when a child is faced with a verbal problem he presumably "sees" the wanted-given structure of that problem. His choice of operation would be based on his recognition of the commonality of that structure's wanted-given attributes with those wanted-given attributes of one of the operations.

Of main concern in the study was the ability of the groups to choose the correct operation to use in solving the types of problems tested. Of lesser interest was the ability to obtain the correct answer and speed in obtaining correct answers. A summary of the results indicated that for all types of problems combined, and for all mental age levels involved (low, medium, high), the "wanted-given" treatment group was found to be superior on all dependent variables measured, i.e., choice of operation, correct answers, and speed. Whether these findings would hold for other types of one-step verbal problems, for two- and three-step problems, and for a wider range of age-grade level children is, of course, not known.

Burns and Yonally (50) attempted to study the effect of varying the order of presentation of numerical data on achievement in two- and three-step verbal arithmetic problems. In other words, if problems are stated with numerical data not given in the order in which they are needed to solve the problem, will pupils solve as many of them successfully as problems stated with numerical data given in the order in which they are used to solve the problems? Using fourth- and fifth-grade students, they concluded that pupils are less successful getting correct answers to word problems when numerical data are presented in some order other than the order in which they will be used to solve the problem. They also found that arithmetic reasoning ability, as measured by a standardized test, is positively related to ability to do problems which present the numerical data in mixed order.

What is the effect of unfamiliarity of setting on verbal problem-solving ability? Brownell (44) reports that, for 65 percent to 80 percent of the children, unfamiliar situations have little effect, but that for 20 percent to 35 percent, unfamiliar settings introduce a new source of difficulty. Lyda and Church (173) also found that the probability of working verbal problems in arithmetic satisfactorily when there has not been direct, practical experience with that particular arithmetic situation is considerably greater for the above-average group of children than the below-average and average; and greater for the average than the below-average.

The teacher of elementary school mathematics can feel quite sure that just giving many verbal problems to students will effect some increment in ability to solve problems. However, as the studies cited suggest, there are specific procedures and techniques that can be utilized that appear to facilitate achievement in verbal problem solving. In many cases the typical textbook program for improving or developing problem-solving ability will have to be supplemented by experiences and techniques, some of which are suggested in the various studies cited in this section. This supplementing of the regular program would seem to be more necessary for students in the lower IQ ranges.

What effect does the teaching of non-decimal numeration systems have on learning of topics in elementary school mathematics?

With the increased emphasis on structure, the basic concepts of a body of knowledge around which it is organized, it has been suggested that the basic properties of the Hindu-Arabic system of numeration come into focus more clearly for students when systems with bases other than ten are taught.

Jackson (142) used fifth- and seventh-grade students in his study of this problem. One treatment group at each grade level was taught a specially prepared unit on non-decimal numeration systems. A second treatment group at each grade level was taught a specially prepared unit on the decimal numeration system. He concluded that the fifth-grade pupils receiving instruction in non-decimal systems indicated greater achievement, as measured by various sub-tests in: understanding of the nature of the decimal system of numeration; understanding of the nature of a numeration system; understanding of the properties of the operations of addition, subtraction, and multiplication; skills in problem solving.

Fifth-grade pupils receiving instruction in the decimal system of numeration did significantly better than pupils receiving instruction in non-decimal systems on a test measuring computational skills in the decimal system. At the seventh-grade level no significant differences in mean scores between the two treatments were found on: understanding of the decimal system of numeration; understanding the properties of addition, subtraction, and multiplication; skill in computation; skill in problem solving. Seventh-grade pupils receiving instruction in non-decimal systems of numeration did significantly better than pupils receiving instruction in the decimal system in mean score on the test measuring an understanding of the nature of a numeration system.

Schlensog (229), using sixth-grade students, supplemented one group's instruction with 13 lessons on numeration systems with various number bases, supplemented a second group's instruction by focusing extra attention upon the decimal system, and made no change in a third group's usual course of arithmetic instruction. Pre- and post-test measuring instruments focused on: understanding of the decimal system and its operations; computational abilities; and changes in preference for arithmetic. Analysis of the data indicated no aspects in which the treatments produced significantly different results.

Two other smaller studies (134, 166), using fourth-grade students, indicated that the study of non-decimal numeration can be carried out at this grade level with some profit for the students.

Critics of teaching other-base systems of numeration in the elementary school, such as Fehr (91), state that:

> All over the world, in every nation, bar none, and in every type of communication—social, business, scientific, professional, etc.—the one system that is used is the decimal system. This is the only system that most (at least 95 percent) of the population will ever use the rest of their lives, and they will probably use it every day of their lives.

The hypothesis that the study of other-base systems will enhance understanding of our own decimal system would seem to be a reasonable justification for its inclusion as a topic for study in the elementary grades. Evidence is not conclusive, however, that this is the only or best way of accomplishing this objective.

The evidence would suggest that the teacher can feel quite confident at this point that some supplementary work in other-base systems of numeration can be done with no evidence of a decrement in learning in other areas of arithmetic which are judged to be of value. Whether there is any advantage in supplementing instruction with other-base numeration instruction over supplementary work with base-ten material is yet unclear.

What about programmed instruction in elementary school mathematics?

Programmed materials and the complementary development of "teaching machines" have experienced a great ground swell of interest as a means of instruction since the last revision of this monograph. A convergence of factors at this particular point in our cultural development has, no doubt, contributed to this surge of interest.

It would be generally accepted that one objective of the schools is to pass on the accumulated knowledge of mankind that has been judged valuable for successive generations. Teachers from the time of Socrates have been aware of the desirability of presenting these various facts, skills, concepts, and generalizations in a step-by-step, sequential manner. Successful textbook writers have used "programming" in their development of topics in various substantive areas continually. Sensitive and successful teachers have also been aware of the importance of motivating and encouraging students in their strivings to learn. This "reinforcement," in some generic sense, is an integral part of most learning theories.

With the quantitative increases and qualitative diversity of an educational system which is attempting quality mass education such as in this country, the possibility of giving individual "reinforcement" becomes more difficult. The sophistication achieved by [illegible] technology, has offered one possible means for achieving some degree of individualization in mass education. The convergence of these various psychological, sociological and technological factors has no doubt contributed to the increased interest in programmed materials and "teaching machines."

It is beyond the scope of this monograph to discuss in detail the philosophical implications, psychological factors, and technical procedures involved with programmed self-instruction. Only studies concerned with elementary school mathematics will be cited. The teacher interested in a broader discussion of the topic may go to a source such as *Teaching Machines and Programmed Learning* (172) or *Teaching by Machine* (254).

Keislar (154), using 14 fifth- and sixth-graders, who were matched on intelligence, sex, reading ability, and pretest scores with a control group, used a multiple-choice programming technique to teach understandings about the area of rectangles. The control group received no instruction on the topic. After 2 to 3 days of work on the program, members of the experimental group were given a post-test on the understandings. All experimental students except one achieved a higher posttest score than did the matched control students. Keislar reported, however, that the experimental group learned far less than was expected from their previous performance in learning.

Studies such as those by Arvin (8) Fincher (93) and Northcutt (189) attempted to compare performance on criterion tests of students who had received "conventional" textbook instruction on a particular topic with the performance of students who had received instruction by means of a programmed self-instruction technique. One of these studies suggested that the programmed self-instructional technique was more effective and more efficient than "conventional" instruction. Another study suggested that there was no difference in effectiveness but that programmed self-instruction was more efficient. Northcutt, meanwhile, found that students receiving teacher instruction made significantly greater gains than did students who worked independently with programmed material.

Another set of studies (7, 73, 189, 198, 275) attempted to delineate some student variables that contribute to successful learning by means of self-instruction techniques. Traweek (275) found with fourth-grade students using a linear program dealing with fractions that:

1. High-gain students showed significantly more test anxiety than low-gain students.

2. High-gain students reported significantly more withdrawn tendencies

3. Low-gain students reported significantly higher self-reliance scores than high-gain students.

4. No significant difference between the mean scores of the high-gain and low-gain students with respect to IQ.

On the other hand, Andrews (7) found that an interaction effect exists between intelligence and content difficulty during programmed instruction. At lower levels of content difficulty all subjects perform with similar adequacy. Also, a relationship was found between sex and mathematics achievement; girls consistently exceed boys in achievement during programmed instruction. Northcutt (189) found after administering a Locus of Control Scale to students that locus of control was not related to achievement for boys, but was significantly correlated with achievement for girls. It was also found that internally controlled subjects completed more items than did externally controlled subjects.

Kalin (152) tried to determine whether programmed instruction could provide intellectually superior fifth- and sixth-grade pupils with opportunity to learn some mathematical concepts normally not taught until secondary school. No significant difference was found between a group taught by a regular elementary teacher using regular methods and a group using a self-instruction method. The latter group learned the same amount of mathematics in 20 percent less time. However, Dessart (75) also found that self-instructional procedures could be used to teach an advanced mathematical concept to superior eighth-grade students.

Blackman and Capobianco (27) compared reading and arithmetic achievement of mentally retarded young adolescents whose average IQ was in the low educable range, using teaching machines and programs, to equated groups taught the same material by "traditional" special class techniques. Results generally indicated that, although both the machine and no-machine groups improved significantly in their reading and arithmetic performance over the school year, no superiority was evidenced for the teaching machine group. Greater improvement in deportment, however, was manifested by the teaching machine groups as compared to the no-teaching machine groups.

Another set of studies explored some of the task variables associated with programmed self-instructional methods. Nelson (188) compared performance and retention of eighth graders on a program dealing with mathematical induction in which the program variations were:

1. Constructed response, fixed sequence
2. Multiple-choice, fixed sequence
3. Multiple-choice, variable sequence.

The three programming techniques used for the development of the concept of mathematical induction were equally effective with regard to post- and retention test results. Austin (11), using sixth-grade students and a program dealing with instruction in multiplication of fractions, found:

> 1. There is no significant difference in the gain score between a text which uses constructed responses and a text which uses multiple-choice responses.
>
> 2. There is no significant difference in the gain score between a text which reinforces 100 percent of the time and one which reinforces 50 percent of the time.

Kaufman (153) compared differences in performance of sixth-grade students when the variable was amount of remedial feedback incorporated into an intrinsically programmed unit on powers and exponents. It was found that neither mastery of material, time to complete material, nor student learning efficiency was significantly affected by the remedial feedback variable under consideration.

CAI, Computer Assisted Instruction, is beginning to receive some attention for use in elementary school mathematics. Suppes *et al.* (263) have been experimenting with using CAI in the drill program at the intermediate grade level. Riedesel and Suydam (212) discuss implications of CAI for teacher education.

In general, it can be said that students using well-written programmed materials will be able to demonstrate improvement increments on appropriate criterion measurements. Superiority of this method over "traditional" teacher-taught methods has not been consistently demonstrated. Much work must be done to determine how programmed self-instructional techniques best fit into the overall instructional program in the elementary schools; also, which students will profit most by use of these self-instructional methods. Programming techniques as well as the most appropriate "hardware" for presenting the programs must also be continually explored. Reflecting on the research cited in this section, one would have to agree with Stolurow, that "the findings are more provocative than definitive."

References

1. Adler, Irving. "Mental Growth and the Art of Teaching." *The Arithmetic Teacher* 13:576-84; November 1966.

2. Alexander, Vincent E. "Sex Difference in Seventh-Grade Problem Solving." *School Science and Mathematics* 42:47-50; January 1962.

3. Almy, Millie. *Young Children's Thinking*. New York: Teachers College

3. Almy, Millie. *Young Children's Thinking*. New York: Teachers College Press, 1966.

4. Amidon, Edmund, and Ned Flanders. "The Effects of Direct and Indirect Teacher Influence on Dependent-Prone Students Learning Geometry." *Journal of Educational Psychology* 52:286-91; 1961.

5. National Council of Teachers of Mathematics. *An Analysis of New Mathematics Programs*. Washington, D. C.: the Council, 1963.

6. Anderson, Lester G. "What the Psychology of Learning Has To Contribute to the Education of Teachers." *Journal of Educational Psychology* 41:362-65; October 1957.

7. Andrews, Mary M. "Programmed Instruction in Elementary Schools: A Study of Variables Associated with Mathematics Achievement." *Dissertation Abstracts* 24:2319; December 1963.

8. Arvin, Charles L. "An Experimental Study of Programmed Instruction in Multiplication of Fractions." *Dissertation Abstracts* 26:7109; June 1966.

9. Atkinson, John W. "The Mainsprings of Achievement-Oriented Activity." In: J. D. Krumboltz, editor. *Learning and the Educational Process*. Chicago: Rand McNally & Company, 1965.

10. Augustine. *The Confessions of Saint Augustine.* Edward B. Pusey, translator. New York: The Modern Library, 1949.

11. Austin, Gilbert R. "A Study of Programmed Instruction Response Styles and Reinforcement Schedules for Teaching Multiplication of Fractions." *Dissertation Abstracts* 26:5218; March 1966.

12. Ausubel, David P. *The Psychology of Meaningful Verbal Learning.* New York: Grune & Stratton, 1963. Reprinted by permission.

13. Ausubel, David P. "How Reversible Are the Cognitive and Motivational Effects of Cultural Deprivation? Implications for Teaching the Culturally Deprived Child." *Urban Education* 1:16-38; Summer 1964.

14. Ausubel, David P. "Review of 'Toward a Theory of Instruction.'" *Harvard Educational Review* 36:337; Summer 1966.

15. Ausubel, David P. "Can Children Learn Anything That Adults Can—And More Efficiently?" *The Elementary School Journal* 62:270-72; February 1962.

16. Baer, C. J. "The School Progress and Adjustment of Underage and Average Students." *Journal of Educational Psychology* 49:17-19; 1958.

17. Bassham, Harrell. "Teacher Understanding and Pupil Efficiency in Mathematics—A Study of Relationship." *The Arithmetic Teacher* 9:383-87; November 1962.

18. Bassham, Harrell; Michael Murphy; and Katherine Murphy. "Attitude and Achievement in Arithmetic." *The Arithmetic Teacher* 11:66-72; February 1964.

19. Baumann, Reemt R. "Children's Understanding of Selected Mathematical Concepts in Grades Two and Four." *Dissertation Abstracts* 26:5219; March 1966.

20. Bean, John E. "Arithmetic Understanding of Elementary School Teachers." *Elementary School Journal* 59:447-50; May 1959.

21. Beckwith, Mary, and Frank Restle. "Process of Enumeration." *Psychological Review* 73:437-44; September 1966.

22. Bereiter, C., and S. Engelmann. "Observations on the Use of Direct Instruction with Young Disadvantaged Children." *Journal of School Psychology* 4:55-62; Spring 1966.

23. Bernstein, Allen. "Library Research—A Study in Remedial Arithmetic." *School Science and Mathematics* 59:185-95; 1959. Reprinted with permission of the editor.

24. Bernstein, Allen L. "Motivations in Mathematics." *School Science and Mathematics* 64:749-54; December 1964. Reprinted by permission of the editor.

25. Biggs, John B. "The Psychopathology of Arithmetic." In: F. W. Land, editor. *New Approaches to Mathematics Teaching.* New York: St. Martin's Press, Inc., 1965.

26. Bjonerod, Corwin E. "Arithmetic Concepts Possessed by the Preschool Child." *The Arithmetic Teacher* 7:347-50; November 1960.

27. Blackman, L. S., and R. J. Capobianco. "Evaluation of Programmed Instruction with the Mentally Retarded Utilizing Teaching Machines." *American Journal of Mental Deficiency* 70:262-69; September 1965.

28. Bodwin, Raymond F. "The Relationship Between Immature Self-Concept and Certain Educational Disabilities." Doctor's dissertation. Lansing: Michigan State University, 1957.

29. Bogut, T. L. "Comparison of Achievement in Arithmetic in England, California, and St. Paul." *The Arithmetic Teacher* 6:87-94; March 1959.

30. Brace, Alec, and D. L. Nelson. "The Pre-School Child's Concept of Number." *The Arithmetic Teacher* 12:126-33; February 1965.

31. Bradley, Betty H., and Marcel Hundziak. "TMI-Grolier Time Telling Program for the Mentally Retarded." *Exceptional Children* 32:17-20; September 1965.

32. Bravo, Ann R. "Formal Preparation for Early Childhood Arithmetic." *The Arithmetic Teacher* 12:56-58; January 1965.

33. Brooke, George M. "The Common Denominator Method in the Division of Fractions." Doctor's dissertation. Iowa City: State University of Iowa, 1954.

34. Brownell, William A. "Arithmetic in Grades I and II: A Critical Summary of New and Previously Reported Research." *Duke University Research Studies in Education, No. 6.* Durham, N. C.: Duke University Press, 1941.

35. Brownell, William A. "Arithmetical Readiness as a Practical Classroom Concept." *Elementary School Journal* 52:15-22; September 1951.

36. Brownell, William A. "Psychological Considerations in the Learning and Teaching of Arithmetic." *The Teaching of Arithmetic.* Tenth Yearbook, National Council of Teachers of Mathematics. Washington, D. C.: the Council, 1935.

37. Brownell, William A. *The Development of Children's Number Ideas in the Primary Grades.* Supplementary Educational Monographs, No. 35. Chicago: University of Chicago Press, August 1928. 241 pp.

38. Brownell, William A. "Arithmetical Abstractions—Progress Toward Maturity of Concepts Under Differing Programs of Instruction." *The Arithmetic Teacher* 10:329; October 1963.

39. Brownell, William A. "The Evaluation of Learning Under Dissimilar Systems of Instruction." *The Arithmetic Teacher* 13:267-74; April 1966.

40. Brownell, William A. "The Evaluation of Learning in Arithmetic." *Arithmetic in General Education.* Sixteenth Yearbook, National Council of Teachers of Mathematics. Washington, D. C.: the Council, 1941. pp. 225-67.

41. Brownell, William A., and Charlotte B. Chazal. "The Effects of Premature Drill in Third-Grade Arithmetic." *Journal of Educational Research* 29:17-28; September 1935.

42. Brownell, William A., and Gordon Hendrikson. "How Children Learn Information Concepts and Generalizations." *Learning and Instruction*. Forty-Ninth Yearbook, National Society for the Study of Education, Part I. Chicago: University of Chicago Press, 1950. pp. 92-128.

43. Brownell, William A., and Harold E. Moser. "Meaningful vs. Mechanical Learning; A Study in Grade III Subtraction." *Duke University Research Studies in Education, No. 8*. Durham, N. C.: Duke University Press, 1949.

44. Brownell, William A., and Lorena B. Stretch. "The Effect of Unfamiliar Settings on Problem Solving." *Duke University Research Studies in Education, No. 1*. Durham, N. C.: Duke University Press, 1931. 86 pp.

45. Brueckner, Leo J. "Diagnosis in Arithmetic." *Educational Diagnosis*. Thirty-Fourth Yearbook, National Society for the Study of Education. Bloomington, Ill.: Public School Publishing Co., 1935. pp. 269-302.

46. Bruner, Jerome S. "The Act of Discovery." *Harvard Educational Review* 31:21-32; 1961.

47. Bruner, Jerome S. *The Process of Education*. Cambridge, Mass.: Harvard University Press, 1960.

48. Bruner, Jerome S. "The Course of Cognitive Growth." *American Psychologist* 19:1; January 1964.

49. Burns, Paul C. "Arithmetic Fundamentals for the Educable Mentally Retarded." *American Journal of Mental Deficiency* 66:5761; 1962. Reprinted by permission.

50. Burns, Paul C., and J. L. Yonally. "Does the Order of Presentation of Numerical Data in Multi-Step Arithmetic Problems Affect Their Difficulty?" *School Science and Mathematics* 64:267-70; April 1964.

51. Burton, William H. *The Guidance of Learning Activities*. New York: Appleton-Century-Crofts, 1952. 737 pp.

52. Buswell, G. T. "A Comparison of Achievement in Arithmetic in England and Central California." *The Arithmetic Teacher* 5:1-9; February 1958.

53. Buswell, G. T. "Methods of Studying Pupils' Thinking in Arithmetic." *Arithmetic 1949*. Supplementary Educational Monographs, No. 70. Chicago: University of Chicago Press, 1949. pp. 58-63.

54. Buswell, G. T., and Lenore Johns. *Vocabulary of Arithmetic*. Chicago: University of Chicago, 1931. 146 pp.

55. Buswell, G. T., and Bert Y. Kersh. *Patterns of Thinking in Solving Problems*. University of California Publications in Education, Vol. 12, No. 2. Berkeley: University of California Press, 1956. pp. 63-148.

56. Callahan, Leroy G. "A Study of Knowledge Possessed by Elementary School Teachers In-Service and In-Training, of the Cultural, Psychological, and Mathematical Foundations of the Elementary School Mathematics Program." Unpublished Ed.D. dissertation. Syracuse, N. Y.: Syracuse University, 1966.

57. Callahan, Leroy G. "Is Your Readiness Program Effective?" *The Instructor* 76:73; February 1967.

58. Callahan, Leroy G. "Remedial Work with Under-Achieving Children." *The Arithmetic Teacher* 9:138-40; March 1962.

59. Capps, Lelon R. "Division of Fractions." *The Arithmetic Teacher* 9:10-16; January 1962.

60. Carlow, Chester. "A Study of Variables Within the Method of Individually Guided Discovery in Secondary School Mathematics: The Experimental Comparison of Conceptual Structures, Consolidation, and Learner Personality with Learning, Retention, and Transfer by Ninth-Grade College Preparatory Males." Ph.D. dissertation. Syracuse, N. Y.: Syracuse University, 1967.

61. Carroll, Marian L. "Academic Achievement and Adjustment of Underage and Average Third Graders." *Journal of Educational Research* 56:415-19; April 1963.

62. Carter, Lowell. "The Effect of Early School Entrance on the Scholastic Achievement of Elementary School Children in the Austin Public Schools." *Journal of Educational Research* 50:91-103; 1956.

63. Clark, John R., and Laura K. Eads. *Guiding Arithmetic Learning.* New York: Harcourt, Brace & World, Inc., 1954.

64. Cleveland, Gerald A. "A Study of Certain Psychological and Sociological Characteristics as Related to Arithmetic Achievement." Ed.D. dissertation, Syracuse, N. Y.: Syracuse University, 1961.

65. Coleman, James S., and others. *Equality of Educational Opportunity.* Washington, D. C.: Superintendent of Documents, U. S. Government Printing Office, 1966. 743 pp.

66. Costello, Helen M. "The Responses of Mentally Retarded Children in Specialized Learning Experiences in Arithmetic." Doctor's dissertation. Philadelphia: University of Pennsylvania, 1941.

67. Crawford, Douglas. "An Investigation of Age-Grade Trends in Understanding the Field Axioms." Doctor's dissertation. Syracuse, N. Y.: Syracuse University, 1964.

68. Cronbach, Lee J. "Issues Current in Educational Psychology." *Monographs of the Society for Research in Child Development*, Vol. 30, No. 1; 1965.

69. Cronbach, Lee J. "The Logic of Experiments on Discovery." In: Shulman and Keislar, editors. *Learning by Discovery: A Critical Appraisal.* Chicago: Rand McNally & Company, pp. 76-92.

70. Crumley, Richard D. "A Comparison of Different Methods of Teaching Subtraction in the Third Grade." Doctor's dissertation. Chicago: University of Chicago, 1956.

71. Crumley, Richard D. "Teaching Rate and Ratio in the Middle Grades." *School Science and Mathematics* 60:143-50; 1960.

72. Cuisenaire, G., and C. Gattegno. *Numbers in Colour.* London: Heinemann, 1954.

73. D'Augustine, Henry. "Factors Relating to Achievement with Selected Topics in Geometry and Topology When Taught to Fifth, Sixth, and Seventh Grade Pupils via a Programmed Text." *Dissertation Abstracts* 24:4538-39; May 1964.

74. Dawson, Dan T. "Number Grouping as a Function of Complexity." *Elementary School Journal* 54:35-42; September 1953.

75. Dessart, Donald J. "A Study of Programmed Learning with Superior Eighth Grade Students." *Dissertation Abstracts* 24:1499-1500; October 1963.

76. Deutsch, Martin. "The Role of Social Class in Language Development and Cognition." *American Journal of Orthopsychiatry* 35:78-87; January 1965. Copyright © 1965, The American Orthopsychiatric Association, Inc. Reprinted by permission.

77. Dickenson, Donald J., and J. Donald Larson. "The Effects of Chronological Age in Months on School Achievement." *Journal of Educational Research* 56:492-93; May 1963.

78. Dienes, Z. P. *Building Up Mathematics*. London: Hutchinson, 1960.

79. Dominy, Mildred M. "A Comparison: Textbooks, Domestic and Foreign." *The Arithmetic Teacher* 10:428-34; November 1963.

80. Dreger, R. M., and L. R. Aiken. "The Identification of Number Anxiety in College Populations." *Journal of Educational Psychology* 48:344-51; December 1957.

81. Dunkley, M. E. "Some Number Concepts of Disadvantaged Children." *The Arithmetic Teacher* 12:359-61; May 1965.

82. Dutton, Wilbur H. "Attitudes of Junior High School Pupils Toward Arithmetic." *School Review* 64:18-22; January 1956.

83. Dutton, Wilbur H. *Evaluating Pupils' Understanding of Arithmetic*. Englewood Cliffs, N. J.: Prentice-Hall, Inc., 1964.

84. Dutton, Wilbur H. "Growth in Number Readiness in Kindergarten Children." *The Arithmetic Teacher* 10:251-55; May 1963.

85. Emm, Sister M. Eloise. *A Factorial Study of the Problem Solving Ability of Fifth Grade Boys*. Washington, D. C.: The Catholic University of America Press, 1959.

86. Engelhard, Sister M. Dominic. *An Experimental Study of Arithmetic Problem Solving Ability of Sixth Grade Girls*. Washington, D. C.: The Catholic University of America Press, 1955.

87. Evans, J. W., Jr. "Effect of Pupil Mobility Upon Academic Achievement." *National Elementary Principal* 45:18-22; April 1966.

88. Faust, Claire E. "A Study of the Relationship Between Attitude and Achievement in Selected Elementary School Subjects." *Dissertation Abstracts* 23:2752-53; February 1963.

89. Fedon, Peter. "The Role of Attitude in Learning Arithmetic." *The Arithmetic Teacher* 5:304-310; December 1958.

90. Fehr, Howard. "Modern Mathematics and Good Pedagogy." *The Arithmetic Teacher* 9:402-411; November 1963.

91. Fehr, Howard. "Sense and Nonsense in a Modern School Mathematics Program." *The Arithmetic Teacher* 13:83-91; February 1966.

92. Feldhusen, J. M. "Anxiety, Divergent Thinking, and Achievement." *Journal of Educational Psychology* 56:40-45; February 1965.

93. Fincher, Glen E. "The Construction and Experimental Application of a Programmed Course on the Addition and Subtraction of Fractions for Grade 5." *Dissertation Abstracts* 24:650; August 1963.

94. Fink, Martin. "Self-Concept as It Relates to Academic Underachievement." *California Journal of Educational Research* 13:57-61; 1962.

95. Finley, Carmen J. "Arithmetic Achievement in Mentally Retarded Children: The Effects of Presenting the Problem in Different Contexts." *American Journal of Mental Deficiency* 67:281-86; September 1962.

96. Fisher, John. "Extent of Implementation of CUPM Level I Recommendations." *The Arithmetic Teacher* 14:194-97; March 1967.

97. Flavell, John. *The Developmental Psychology of Jean Piaget*. Princeton, N. J.: D. Van Nostrand Company, Inc., 1963.

98. Foshay, Arthur W. "From the Association." *Balance in the Curriculum*. 1961 Yearbook. Washington, D. C.: Association for Supervision and Curriculum Development, 1961. pp. iii-iv.

99. Fouty, Arthur T. "A Study of the Effects of Mobility and Related Factors on the Academic Achievement of Children in a Suburban School." *Dissertation Abstracts* 25:3351; December 1964.

100. Friedlander, B. "A Psychologist's Second Thoughts on Concepts, Curiosity, and Discovery in Teaching and Learning." *The Harvard Educational Review* 35:18-38; 1965.

101. Gagne, Robert. *The Conditions of Learning*. New York: Holt, Rinehart, & Winston, Inc., 1965. p. 168.

102. Gagne, R. M.; J. R. Mayor; H. L. Garstens; and N. E. Paradise. "Factors in Acquiring Knowledge of a Mathematical Task." *Psychological Monographs* 76(7); 1962. (Whole No. 526.)

103. Gallagher, James J. "Analysis of Research on the Education of Gifted Children." Springfield, Ill.: State of Illinois, Office of the Superintendent of Public Instruction, 1960.

104. Galperin, P. Ia. "An Experimental Study in the Formation of Mental Actions." In: B. Simon, editor. *Psychology in the Soviet Union*. Stanford, California: Stanford University Press, 1957. pp 217, 222-23.

105. Gibb, E. Glenadine. "Children's Thinking in the Process of Subtraction." *Journal of Experimental Education* 25:71-80; September 1965.

106. Gilmary, Sister M. "Examination of Some of Piaget's Principles in Application to Psychology of Arithmetic." *Catholic Education Review* 62:369-75; September 1964.

107. Glennon, Vincent J. " . . . And Now Synthesis: A Theoretical Model for Mathematics Education." *The Arithmetic Teacher* 12:134-41; February 1965.

108. Glennon, Vincent J. "A Study in Needed Redirection in the Preparation of Teachers of Arithmetic." *The Mathematics Teacher* 42:389-96; December 1949.

109. Glennon, Vincent J. "Some Perspectives in Education." *Enrichment Mathematics for the Grades.* The Twenty-Seventh Yearbook. Washington, D. C.: National Council of Teachers of Mathematics, 1963.

110. Glennon, Vincent J. "Testing Meanings in Arithmetic." *Arithmetic 1949.* Supplementary Educational Monographs, No. 70. Chicago: University of Chicago Press, November 1949. pp. 64-74.

111. Glennon, Vincent J., and John W. Wilson. "Developing Effective Evaluation Procedures." In: M. Vere DeVault, editor. *Improving Mathematics Programs.* Columbus, Ohio: Charles E. Merrill Books, Inc., 1961.

112. *Goals for School Mathematics.* The Report of the Cambridge Conference on School Mathematics. Boston: Educational Development Center, Inc. and Houghton Mifflin Company, 1963.

113. Goldberg, Miriam L.; A. Harry Passow; and Joseph Justman. *The Effect of Ability Grouping.* New York: Teachers College Press, 1966. Reprinted by permission of the publisher. Copyright © 1966, Teachers College, Columbia University.

114. Goldstein, Avram. "Does Homework Help? A Review of Research." *The Elementary School Journal* 60:213-24; January 1960.

115. Gordon, Edmund W. "Characteristics of Socially Disadvantaged Children; Summary and Outlook." *Review of Educational Research* 35:384-85; December 1965.

116. Gott, Margaret. "The Effect of Age Differences at Kindergarten Entrance on Achievement and Adjustment in the Elementary School." *Dissertation Abstracts* 24:3640; March 1964.

117. Graubard, P. S. "Extent of Academic Retardation in a Residential Treatment Center." *Journal of Educational Research* 58:78-80; October 1964.

118. Gray, Roland F. "An Experiment in the Teaching of Introductory Multiplication." *The Arithmetic Teacher* 12:199-203; March 1965.

119. Greenfield, Patricia Marks. "On Culture and Conservation." In: Jerome S. Bruner, Rose Olver, and Patricia M. Greenfield, *et al. Studies in Cognitive Growth.* New York: John Wiley & Sons, Inc., 1966 pp. 225-56.

120. Gruen, G. E. "Experiences Affecting the Development of Number Conservation in Children." *Child Development* 36:963-79; December 1965.

121. Gunderson, A. G. "Thought Patterns of Young Children in Learning Multiplication and Division." *Elementary School Journal* 55:453-61; April 1953.

122. Haggard, Ernest A. "Specialization, Personality, and Academic Achievement in Gifted Children." *The School Review* 65:388-414; Winter 1957.

123. Hansen, Carl. "Factors Associated with Successful Achievement in Problem Solving in Sixth Grade Arithmetic." *Journal of Educational Research* 38:15; October 1944.

124. Harper, Harold E. "Elementary Teacher's Knowledge of Basic Arithmetic Concepts and Symbols." *The Arithmetic Teacher* 11:543-46; December 1964.

125. Hartung, Maurice. "Formalism in Arithmetic Programs." *The Arithmetic Teacher* 9:371-75; November 1962.

126. Haskell, Simon. "Some Observations on the Effects of Class Size Upon Pupil Achievement in Geometrical Drawing." *Journal of Educational Research* 58:27-30; September 1964.

127. Hebron, M. E. "A Factorial Study of Learning a New Number System and Its Relation to Attainment, Intelligence, and Temperament." *British Journal of Educational Psychology* 32:38-45; February 1962.

128. Heddens, James, and Kenneth Smith. "The Readability of Elementary Mathematics Books." *The Arithmetic Teacher* 11:466-68; November 1964.

129. Hendrix, Gertrude. "Learning by Discovery." *The Mathematics Teacher* 54:290-99; 1961.

130. Hess, Eckhard H. "Attitude and Pupil Size." *Scientific American* 212:46-54; April 1965.

131. Hilgard, Ernest. *Theories of Learning*. New York: Appleton-Century-Crofts, 1956.

132. Hill, Edwin H. "A Study of Preference and Performance on Partition and Measurement Division Problems." Unpublished doctoral thesis. Iowa City: State University of Iowa, 1952.

133. Hlavaty, Julius H., editor. *Mathematics for the Academically Talented Student in the Secondary School*. Washington, D. C.: National Council of Teachers of Mathematics, 1959.

134. Hollis, Loye Y. "Why Teach Numeration?" *The Arithmetic Teacher* 11:94-97; February 1964.

135. Holton, Boyd. "Motivation and General Mathematics Students." *The Mathematics Teacher* 57:20; January 1964.

136. Houston, Robert W., and M. Vere DeVault. "Mathematics In-Service Education: Teacher Growth Increases Pupil Growth." *The Arithmetic Teacher* 10:243-47; May 1963.

137. Hudgins, B., and L. Loftis. "The Invisible Child in the Arithmetic Class: A Study of Teacher-Pupil Interaction." *Journal of Genetic Psychology* 108:143-52; 1966.

138. Hudson, James A. "A Pilot Study of the Influence of Homework in

Seventh Grade Mathematics and Attitude Toward Homework in the Fayetteville Public Schools." *Dissertation Abstracts* 26:906; August 1965.

139. Husén, Torsten, editor. *International Study of Achievement in Mathematics: A Comparison of Twelve Countries.* New York, N. Y.: John Wiley & Sons, Inc., 1967. Vol. I, Vol. II.

140. Ilika, Joseph. "Age of Entrance into the First Grade as Related to Scholastic Achievement." *Dissertation Abstracts* 24:2737; January 1964.

141. Irish, Elizabeth H. "Improving Problem Solving by Improving Verbal Generalization." *The Arithmetic Teacher* 11:196-75; April 1964.

142. Jackson, Robert L. "Numeration Systems: An Experimental Study of Achievement on Selected Objectives of Mathematics Education Resulting from the Study of Different Numeration Systems." *Dissertation Abstracts* 26:5292-93; March 1966.

143. Jacobs, James N.; Althea Beery; and Judith Lemwohl. "Evaluation of an Accelerated Arithmetic Program." *The Arithmetic Teacher* 12:113-19; February 1965.

144. Jarvis, O. T. "Boy-Girl Ability Differences in Elementary School Arithmetic." *School Science and Mathematics* 64:657-59; November 1964. Reprinted by permission of the editor.

145. Johnson, John T. "Decimal Versus Common Fractions." *The Arithmetic Teacher* 201-203; November 1956.

146. Johnson, John T. "On the Nature of Problem-Solving in Arithmetic." *Journal of Educational Research* 43:110-15; October 1949.

147. Johnson, John T. "Relative Merits of Three Methods of Subtraction." *Contributions to Education,* No. 738. New York: Bureau of Publications, Teachers College, Columbia University, 1938.

148. Josephina, Sister M. "A Study of Attitudes in the Elementary Grades." *Journal of Educational Sociology* 37:56; October 1959.

149. Junge, Charlotte W. "Gifted Children and Arithmetic." *The Arithmetic Teacher* 4:141-46; October 1957.

150. Kagan, Jerome. "Learning, Attention, and the Issue of Discovery." In: *Learning by Discovery: A Critical Appraisal,* Shulman and Keislar, editors. Chicago: Rand McNally & Company, 1966. pp. 151-61.

151. Kagan, Jerome; Leslie Pearson; and Lois Welch. "Conceptual Impulsivity and Inductive Reasoning." *Child Development* 37:385-94; September 1966.

152. Kalin, Robert. "Development and Evaluation of a Programmed Test in an Advanced Mathematical Topic for Intellectually Superior 5th and 6th Grade Pupils." *Dissertation Abstracts* 22:3116-17; March 1962.

153. Kaufman, R. A. "An Experimental Evaluation of Operating Characteristics of an Intrinsic Program on Powers and Exponents of Numbers." *Dissertation Abstracts* 25:1007; August 1964.

154. Keislar, Evan R. "The Development of Understanding in Arithmetic by a Teaching Machine." *Journal of Educational Psychology* 50:247-53; 1959.

155. Kemp, C. Gratton. "Effect of Dogmatism on Critical Thinking." *School Science and Mathematics* 60:314-19; 1960.

156. Kennedy, Wallace, and John Walsh. "A Factor Analysis of Mathematics Giftedness." *Psychological Reports* 17:115-19; 1965. Reprinted by permission of the author and publisher.

157. Kenney, Russell A. "Mathematical Understandings of Elementary School Teachers." *The Arithmetic Teacher* 12:431-42; October 1965.

158. Kersh, Robert. "Learning by Discovery: What Is Learned?" *The Arithmetic Teacher* 11:226-31; 1964.

159. Kessen, W. "Discussion of 'Learning and Using the Mathematical Concept of a Function.'" "Mathematical Learning." *Monograph of the Society for Research in Child Development.* Vol. 30, No. 1. 1965. Copyright © 1965, Society for Research in Child Development, Inc.

160. Kirk, Samuel J., and G. Orville Johnson. *Educating the Retarded Child.* Boston: Houghton Mifflin Company, 1951. 434 pp.

161. Kliebhan, Sister M. Camille. *An Experimental Study of Arithmetic Problem Solving Ability of Sixth Grade Boys.* Washington, D. C.: The Catholic University of America Press, 1955.

162. Kline, Morris. "Intellectuals and the Schools: A Case History." *Harvard Educational Review* 36:505-11. Fall 1966.

163. Koch, Elmer A. "Homework in Arithmetic." *The Arithmetic Teacher* 12:9-13; January 1965.

164. Kramer, Klaas. "Comparison of Objectives, Methods, and Achievement in the United States and in the Netherlands." Doctor's dissertation. Iowa City: University of Iowa, 1957.

165. Lavin, David E. *The Prediction of Academic Performance.* New York: Russell Sage Foundation, 1965.

166. Lerch, H. H. "Fourth-Grade Pupils Study a Number System with Base 5." *Journal of Educational Research* 57:59-63; October 1963.

167. Levy, David M. *Maternal Over-Protection.* New York: Columbia University Press, 1943.

168. Lindgren, H. C., *et al.* "Attitudes Toward Problem Solving as a Function of Success in Arithmetic in Brazilian Elementary Schools." *Journal of Educational Research* 58:44-45; September 1964.

169. Lovell, K. *The Growth of Basic Mathematical and Scientific Concepts in Children.* London: University of London Press, 1962.

170. Luchins, A. S. "Classroom Experiments on Mental Set." *American Journal of Psychology* 59:295-98; 1964.

171. Lucow, William A. "Testing the Cuisenaire Method." *The Arithmetic Teacher* 20:435-38; November 1963.

172. Lumsdaine, A. A., and R. Glaser, editors. *Teaching Machines and Programmed Learning: A Source Book.* Washington, D. C.: National Education Association, 1960.

173. Lyda, W. J., and R. S. Church. "Direct, Practical Arithmetical Experiences and Success in Solving Realistic Verbal Reasoning Problems in Arithmetic." *Journal of Educational Research* 57:530-33; July 1964.

174. Martin, Mavis D. "Reading Comprehension, Abstract Verbal Reasoning, and Computation, as Factors in Arithmetic Problem Solving." *Dissertation Abstracts* 24:4547-48; May 1964.

175. McCandless, Boyd R., and Alfred Castaneda. "Anxiety in Children, School Achievement, and Intelligence." *Child Development* 27:379-82; September 1956.

176. McClelland, David C., *et al. The Achievement Motive.* New York: Appleton-Century-Crofts, 1953.

177. McGowan, Eugene. "Anxiety, Reality of Aspiration, and the Academic Achievement of 4th Grade Children." Ph.D. dissertation. New York: New York University, 1960.

178. McLellan, James O., and John Dewey. *The Psychology of Number.* New York: D. Appleton & Co., 1960.

179. Melton, Arthur W. "The Science of Learning and the Technology of Educational Method." *Harvard Educational Review* 29:96-106; Spring 1959.

180. Mennite, Daniel J. "A Study of the Relationship Between Class Size and Pupil Achievement in the Catholic Elementary School." *Dissertation Abstracts* 25:2854-55; November 1964.

181. Milliken, R. L. "Mathematical-Verbal Ability Differentials of Situational Anxiety as Measured by Blood Pressure Change." *Journal of Experimental Education* 32:309-11; September 1964.

182. Milliken, Robert L., and Bernard Spelka. "Mathematical Verbal Ability Differentials and Somatic Expressions of Situational Anxiety." *Journal of Experimental Education* 31:3-26; September 1962.

183. Montague, David. "Arithmetic Concepts of Kindergarten Children in Contrasting Socio-Economic Areas." *Elementary School Journal* 64:393-97; April 1964.

184. Montessori, Maria. *The Montessori Method.* New York: Schocken Books, Inc., 1964.

185. Mueller, Francis J. "On the Fraction as a Numeral." *The Arithmetic Teacher* 8:234-38; May 1961.

186. Mullins, William H. "An Arithmetic Program for the Superior Student." *Dissertation Abstracts.* Vol. 20. Ph.D. dissertation. Iowa City: University of Iowa, 1958.

187. Neill, A. S. *Summerhill: A Radical Approach to Child Rearing.* New York: Hart Publishing Company, 1960.

188. Nelson, Charles. "A Comparison of Three Programming Techniques

for the Development of the Concept of Mathematical Induction with 8th Grade Students." *Dissertation Abstracts* 23:1627; November 1962.

189. Northcutt, Mary P. "The Comparative Effectiveness of Classroom and Programmed Instruction in the Teaching of Decimals to 5th Grade Students." *Dissertation Abstracts* 24:5091-92; June 1964.

190. Orleans, J. S., and E. Wandt. "The Understanding of Arithmetic Possessed by Teachers." *Elementary School Journal* 53:501-507; May 1953.

191. Overholt, Elbert D. "A Piagetian Conservation Concept." *The Arithmetic Teacher* 12:317-25; May 1965.

192. Pace, Angela. "Understanding and the Ability To Solve Problems." *The Arithmetic Teacher* 8:226-32; May 1961.

193. Pace, Angela. "Understanding of Basic Concepts of Arithmetic: A Comparative Study." *The Journal of Educational Research* 60:107-20; November 1966.

194. Parsley, Kenneth M. "Further Investigation of Sex Differences in Achievement of Under-Average and Over-Achieving Students Within Five IQ Groups in Grades Four Through Eight." *Journal of Educational Research* 57:268-70; January 1964.

195. Passy, Robert A. "Socio-Economic Status and Mathematics Achievement." *The Arithmetic Teacher* 11:469-70; November 1964.

196. Passy, Robert A. "The Effect of Cuisenaire Materials on Reasoning and Computation." *The Arithmetic Teacher* 20:439-40; November 1963.

197. Perrodin, A. F., and W. T. Snipes. "Relationship of Mobility to Achievement in Reading Arithmetic and Languages in Selected Georgia Elementary Schools." *Journal of Educational Research* 59:315-19; March 1966.

198. Pikaart, Leonard. "A Factor Analytic Study of Success in Programmed Textbook Instruction in Elementary School Arithmetic." *Dissertation Abstracts* 24:2383-84; December 1963.

199. Philips, Berman N. "Sex, Social Class, and Anxiety as Sources of Variation in School Achievement." *Journal of Educational Psychology* 53:316-22; December 1962.

200. Phillips, Jo. "Basic Laws for Young Children." *The Arithmetic Teacher* 12:525-32; November 1965.

201. Piaget, Jean. *The Child's Conception of Number*. New York: W. W. Norton and Company, Inc., 1965. p. 3.

202. Piaget, Jean, *et al. The Child's Conception of Geometry*, New York: Harper & Row, Publishers, Inc., 1964.

203. Piaget, Jean. "The Genetic Approach to the Psychology of Thought." *Journal of Educational Psychology* 52:275-81; December 1961.

204. Pines, Maya. "A Pressure Cooker for Four-Year-Old Minds." *Harper's Magazine* 234:55-61; January 1967.

205. Plank, Emma N. "Observations on Attitudes of Young Children Toward Mathematics." *The Mathematics Teacher* 43:252-63; October 1950.

206. Poffenberger, Thomas, and Donald A. Norton. "Factors Determining Attitudes Toward Arithmetic and Mathematics." *The Arithmetic Teacher* 3:113-16; April 1956.

207. Poffenberger, Thomas, and Donald Norton. "Factors in the Formation of Attitudes Toward Mathematics." *Journal of Educational Research* 52(5): 171-76; January 1959.

208. Prehm, H. J. "Concept Learning in Culturally Disadvantaged Children as a Function of Verbal Pretraining." *Exceptional Children* 32:599-604; May 1966.

209. Reed, Mary K. "Vocabulary Load of Certain State-Adopted Mathematics Textbooks, Grades 1-3." *Dissertation Abstracts* 26:3706; January 1966.

210. "Report to the Board of Education on the Class Size Experiment." California: San Diego City Schools, Elementary Schools Division, April 13, 1965. (Mimeographed.)

211. Repp, Florence C. "The Vocabularies of Five Recent Third Grade Arithmetic Textbooks." *The Arithmetic Teacher* 7:128-32; 1960.

212. Riedesel, C. Alan, and Marilyn N. Suydam. "Computer-Assisted Instruction: Implications for Teacher Education." *The Arithmetic Teacher* 14:24-29; January 1967.

213. Riess, Anita P. "New Approach to the Teaching of Fractions in the Intermediate Grades." *School Science and Mathematics* 64:111-19; February 1964.

214. Riess, Anita P. "The New Arithmetic and 'Abstraction': A Critical View." *School Science and Mathematics* 65:409-15; May 1965.

215. Robinson, Arthur E. *The Professional Education of Elementary Teachers in the Field of Arithmetic.* New York: Bureau of Publications, Teachers College, Columbia University, 1936. p. 67.

216. Rose, Alvin W., and Helen C. Rose. "Intelligence, Sibling Position, and Socio-Cultural Background as Factors in Arithmetic Performance." *The Arithmetic Teacher* 8:50-56; February 1961.

217. Rosenbloom, Paul C. "Implications of Psychological Research." In: *The Low Achiever in Mathematics.* Washington, D. C.: U. S. Department of Health, Education, and Welfare, Bulletin 31, 1965.

218. Ross, Ramon. "A Description of Twenty Arithmetic Underachievers." *The Arithmetic Teacher* 11:235-41; April 1964.

219. Rowland, Monroe, and James Inskeep. "Subject Preferences of Upper Elementary School Children in Cajon Valley Union School District." *California Journal of Educational Research* 14:189; September 1963.

220. Ruch, G. M., and Cyrus D. Mead. "A Review of Experiments on Subtraction." Report of the Society's Committee on Arithmetic, the Twenty-Ninth Yearbook of the NSSE. Bloomington, Illinois: Public School Publishing Co., 1930. pp. 671-78.

221. Ruddell, A. K., and I. H. Balow. "Evaluation of an In-Service Program." *Journal of Educational Research* 57:34-38; September 1963.

222. Ruddell, A. K., and G. W. Brown. "In-Service Education in Arithmetic: Three Approaches." *Elementary School Journal* 64:377-82; April 1964.

223. Ruebush, Britton K. "Anxiety!" *Child Psychology.* Sixty-Second Yearbook of the National Society for the Study of Education, Part I. Chicago: University of Chicago Press, 1963.

224. Sarason, Seymour B., *et al.* "A Longitudinal Study of the Relation of Test Anxiety to Performance on Intelligence and Achievement Tests." *Monograph Society for Research in Child Development.* Vol. 29; No. 7; Serial No. 98; 1964.

225. Sarason, Seymour B., *et al.* "Anxiety in Elementary School Children." New York: John Wiley & Sons, Inc., 1960.

226 Sarnoff, Irving, *et al.* "Test Anxiety and the 'Eleven Plus' Examinations." *British Journal of Educational Psychology* 29:9-16; 1959.

227. Schell, Leo M. "Two Aspects of Introductory Multiplication: The Array and the Distributive Property." *Dissertation Abstracts* 25:5161-62; March-April 1965.

228. Schell, L. M., and P. C. Burns. "Pupil Performance with Three Types of Subtraction Situations." *School Science and Mathematics* 62:208-14; March 1962. Reprinted by permission of the editor.

229. Schlensog, George W. "The Effects of Supplementing Sixth Grade Arithmetic Instruction with a Study of Other Number Bases." *Dissertation Abstracts* 26:5307; March 1966.

230. *School Mathematics Study Group. Newsletter No. 15.* Stanford, California: Stanford University, April 1963.

231. Schroeder, L. B. "Study of the Relationship Between 5 Descriptive Categories of Emotional Disturbance and Reading and Arithmetic Achievement." *Exceptional Children* 32:111-12; October 1965.

232. Scott, Lloyd. "A Study of Teaching Division Through the Use of Two Algorisms." *School Science and Mathematics* 63:739-52; 1963.

233. Sears, Pauline, and Ernest Hilgard. *Theories of Learning and Instruction.* Sixty-Third Yearbook. Part I. National Society for the Study of Education. Chicago: University of Chicago Press, 1964.

234. Shane, Harold G. "Grouping Practices Seem To Favor Composite Plan." *Nation's Schools* 49:72-73; May 1952. Reprinted by permission, from *The Nation's Schools.* Copyright © 1952, The Modern Hospital Publishing Co., Inc., Chicago. All rights reserved.

235. Shapiro, Esther. "Attitudes Toward Arithmetic Among Public School Children in the Intermediate Grades." Ph.D. dissertation. Denver: University of Denver, 1961.

236. Shaw, Merville C., and Gerald J. Alves. "The Self-Concept of Bright Academic Underachievers." *Personnel and Guidance Journal* 42:401-403; December 1963.

237. Shipp, Donald E., and George H. Deer. "The Use of Class Time in Arithmetic." *The Arithmetic Teacher* 7:117-21; March 1960.

238. Shulman, Lee, and Evan Keislar, editors. *Learning by Discovery: A Critical Appraisal.* Chicago: Rand McNally & Company, 1966.

239. Shuster, Albert H., and Fred L. Pigge. "Retention Efficiency of Meaningful Teaching." *The Arithmetic Teacher* 12:24-31; January 1965.

240. Simon, H. A. "An Information-Processing Theory of Intellectual Development." In: W. Kessen, and C. Kuhlman. *Thought in the Young Child.* Monograph of Society for Research on Child Development 27:150-55; 1962.

241. Sizer, Theodore R. "Classroom Revolution: Reform Movement or Panacea?" *The Saturday Review* 48:52-54; June 19, 1965.

242. Slavina, L. S. "Specific Features on the Intellectual Work of Unsuccessful Pupils." In: B. Simon, editor. *Psychology in the Soviet Union.* Stanford, California: Stanford University Press, 1967. p. 205.

243. Sluser, Theodore. "A Comparative Study of Division of Fractions in Which an Explanation of the Reciprocal Principle Is the Experimental Factor." *Dissertation Abstracts* 23; June 1963.

244. Smedlund, J. "The Acquisition of Conservation of Substance and Weight in Children. VI Practice on Continuous versus Discontinuous Material in Conflict-Situations Without External Reinforcement." *Scandinavian Journal of Psychology* 2:203-10; 1961.

245. Smith, Gary. "Commentary upon Suppes-Binford Report of Teaching Mathematical Logic to Fifth- and Sixth-Grade Pupils." *The Arithmetic Teacher* 13:640-43; December 1966.

246. Smith, Kenneth J., and James W. Heddens. "The Readability of Experimental Mathematics Materials." *The Arithmetic Teacher* 11:391-94; October 1964.

247. Spitzer, Herbert F. "Procedures and Techniques for Evaluating the Outcomes of Instruction in Arithmetic." *Arithmetic 1948.* Supplementary Educational Monographs, No. 66. Chicago: University of Chicago Press, October 1948; pp. 15-25.

248. Stauffer, R. G. "Vocabulary Study Comparing Reading, Arithmetic, Health, and Science Texts." *The Reading Teacher* 20:141-47; November 1966. Reprinted with permission of the International Reading Association and R. G. Stauffer.

249. Stephens, Lois E. "What Concepts of Telling Time Can Be Developed by Kindergarten Children?" *Dissertation Abstracts* 25:1793-94; September 1964.

250. Stephens, Lois E. "A Comparative Study of Procedures Used in Teaching Division of Fractions in Sixth Grade Arithmetic." Unpublished master's thesis. University of California at Los Angeles, 1957.

251. Stephens, Lois E. "Retention of the Skill of Division of Fractions." *The Arithmetic Teacher* 7:28-31; January 1960.

252. Stern, Catherine. *Children Discover Arithmetic.* New York: Harper & Row, Publishers, Inc., 1949.

253. Stevenson, H. W., and R. D. Odom. "Relation of Anxiety to Children's Performance on Learning and Problem-Solving Tasks." *Child Development* 36:1003-1012; December 1965.

254. Stolurow, Lawrence. "Teaching by Machine." Cooperative Research Monograph No. 6. Washington, D. C.: U. S. Department of Health, Education, and Welfare. 1961.

255. Stone, Marshall H. "Goals for School Mathematics: A Book Review." *The Mathematics Teacher* 58:353-60; April 1965.

256. Sueltz, Ben A., *et al.* "The Measurement of Understanding in Elementary School Mathematics." *The Measurement of Understanding.* Forty-Fifth Yearbook, Part I. National Society for the Study of Education. Chicago: University of Chicago Press, 1946. pp. 138-56.

257. *Summer Education for Children of Poverty.* Report of the National Advisory Council on the Education of Disadvantaged Children. Washington, D. C.: U. S. Department of Health, Education, and Welfare, 1966.

258. Suppes, Patrick. "A Reply to Gary R. Smith." *The Arithmetic Teacher* 14:635; December 1967.

259. Suppes, Patrick. "Accelerated Program in Elementary School Mathematics—the Second Year." *Psychology in the Schools* 3:294-307; October 1966.

260. Suppes, Patrick. *Sets and Numbers.* Teacher's Edition. New York: The L. W. Singer Company, Inc., 1967.

261. Suppes, Patrick, and Frederick Binford. "Experimental Teaching of Mathematical Logic in the Elementary School." *The Arithmetic Teacher* 12(3):187-95; March 1965.

262. Suppes, Patrick, and Duncan Hansen. "Accelerated Program in Elementary School Mathematics—The First Year." *Psychology in the Schools* 2:195-203.

263. Suppes, Patrick; Max Jerman; and Guy Groen. "Arithmetic Drills and Review on a Computer-Based Teletype." *The Arithmetic Teacher* 13:303-309; April 1966.

264. Sussman, David. "Number Readiness of Kindergarten Children." Unpublished doctoral dissertation. University of California at Los Angeles, 1962.

265. Swenson, Esther J. "Arithmetic for Preschool and Primary-Grade Children." *The Teaching of Arithmetic.* Fiftieth Yearbook, Part II, National Society for the Study of Education. Chicago: University of Chicago Press, 1951, pp. 53-75.

266. The National Council of Teachers of Mathematics. *Evaluation in Mathematics.* Twenty-Sixth Yearbook. Washington, D. C.: the Council, 1961.

267. *The Scottish Council for Research in Education: Its Aims and Activities.* A Publication of the Scottish Council for Research in Education. Number 24. Revised Edition. London: University of London Press, 1953. pp. 18-19, 23.

268. "The Training of Elementary School Mathematics Teachers. An Abridgment of the Recommendations of the Mathematical Association of America for the Training of Teachers of Mathematics." *The Arithmetic Teacher* 7:421-25; December 1960.

269. Thomason, G. M., and A. F. Perrodin. "Comparison of Arithmetic Achievement in England, Central California, and Georgia." *The Arithmetic Teacher* 11:181-85; March 1964.

270. Thompson, George; Eric Gardner; and Francis Di Vesta. *Educational Psychology.* New York: Appleton-Century-Crofts, 1959.

271. Todd, Robert. "A Mathematics Course for Elementary Teachers: Does It Improve Understanding and Attitude?" *The Arithmetic Teacher* 13:198-202; March 1966.

272. Torrance, E. Paul. "Conditions of Creative Learning." *Childhood Education* 39:367; 1963.

273. Tracy, N. H. "Comparison of Test Results—North Carolina, California, and England." *The Arithmetic Teacher* 6:199-202; October 1959.

274. Treacy, John P. "The Relationship of Reading Skills to the Ability To Solve Arithmetic Problems." *Journal of Educational Research* 38:94; October 1944.

275. Traweek, Melvin W. "The Relationship Between Certain Personality Variables and Achievement Through Programmed Instruction of Fourth-Grade Arithmetic." *Dissertation Abstracts* 25:1762; September 1964.

276. Unkel, Esther. "A Study of the Interaction of Socio-Economic Groups and Sex Factors with the Discrepancy Between Anticipated Achievement and Actual Achievement in Elementary School Mathematics." Ed.D. dissertation. Syracuse: Syracuse University, 1965.

277. Vanderlinde, L. F. "Does the Study of Quantitative Vocabulary Improve Problem Solving?" *Elementary School Journal* 65:143-52; December 1964.

278. Van Engen, Henry. "Rate Pairs, Fractions, and Rational Numbers." *The Arithmetic Teacher* 7:389-99; December 1960.

279. Van Engen, Henry, and E. Glenadine Gibb. *General Mental Functions Associated with Division.* Educational Service Studies, No. 2. Cedar Falls: Iowa State Teachers College, 1956. 181 pp.

280. Wallach, Lise, and Richard L. Spratt. "Inducing Number Conservation in Children." *Child Development* 35:1057-71; December 1954.

281. Weatherman, Richard F. "A Study of the Relationships of Mobility to Academic Self-Concept and Academic Achievement of Sixth-Grade Children." *Dissertation Abstracts* 25:2357-58; October 1964.

282. Weaver, Fred J. "A Crucial Problem in the Preparation of Elementary School Teachers." *Elementary School Journal* 56:255-61; February 1956.

283. Weaver, J. Fred, and Cleo F. Brawley. "Enriching the Elementary School Mathematics Program for More Capable Children." *Journal of Education* 142:1-40; October 1959.

284. Weber, C. A. "Do Teachers Understand Learning Theory?" *Phi Delta Kappan* 46:433-35; May 1965.

285. Whitehead, Alfred North. *The Aims of Education*. New York: The Macmillan Company, 1929.

286. Williams, Alfred H. "Mathematical Concepts, Skills, and Abilities of Kindergarten Entrants." *The Arithmetic Teacher* 12:261-68; April 1965.

287. Wilson, Guy M. *Teaching the New Arithmetic*. New York: McGraw-Hill Book Company, Inc., 1951. Copyright © 1951, McGraw-Hill Book Company, Inc. Used by permission.

288. Wilson, John W. "Diagnosis and Treatment in Mathematics: Its Progress, Problems and Potential Role in Educating Emotionally Disturbed Children and Youth." In: Peter Knoblock, and John L. Johnson, editors. *The Teaching-Learning Process in Educating Emotionally Disturbed Children*. New York: Syracuse University Press, 1967.

289. Wilson, John W. "The Role of Structure in Verbal Problem-Solving in Arithmetic: An Analytical and Experimental Comparison of Three Problem-Solving Programs." Ph.D. dissertation. Syracuse: Syracuse University, 1964.

290. Kessen, W., and C. Kuhlman. *Thought in the Young Child*. Monograph of Society for Research in Child Development 27:150-55; 1962.

291. Wohlwill, Joachim F. "A Study of the Development of the Number Concept by Scalogram Analysis." *The Journal of Genetic Psychology* 87:345-77; 1960.

292. Wohlwill, Joachim F. "Piaget's Theory of the Development of Intelligence in the Concrete-Operations Period." In: "Cognitive Models and Development in Mental Retardation." Monograph Supplement to *American Journal of Mental Deficiency*, Vol. 70, No. 4; January 1966.

293. Woolcock, Cyril W. *New Approaches to the Education of the Gifted*. Morristown, N. J.: Silver Burdett Company, 1961. 112 pp.

294. Wright, E.; J. Muriel; and Virginia H. Procter. "Systematic Observation of Verbal Interaction as a Method of Comparing Mathematics Lessons." St. Louis, Missouri: Washington University. U. S. Office of Education Cooperative Research Project No. 816; 1961.

295. Zweibelson, Irving, and Francis Lodato. "Relationship of Pupil Anxiety and Attitude to Arithmetic Readiness and Achievement." *Psychology in the Schools* 2:140-42; April 1965.

296. Zweng, Marylin J. "Division Problems and the Concept of Rate." *The Arithmetic Teacher* 8:547-56; December 1964.

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